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127

Victor Isakov

# Inverse Problems for Partial Differential Equations

Second Edition

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# Inverse Problems for Partial Differential Equations

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To my wife Julie

Most people, if you describe a train of events to them, will tell you what the result would be. They can put those events together in their minds, and argue from them that something will come to pass. There are few people, however, who, if you told them a result, would be able to evolve from their own inner consciousness what the steps were which led up to that result. This power is what I mean when I talk of reasoning backward, or analytically.

—Arthur Conan Doyle, *A Study in Scarlet*

# Preface to the Second Edition

In 8 years after publication of the first version of this book, the rapidly progressing field of inverse problems witnessed changes and new developments. Parts of the book were used at several universities, and many colleagues and students as well as myself observed several misprints and imprecisions. Some of the research problems from the first edition have been solved. This edition serves the purposes of reflecting these changes and making appropiate corrections. I hope that these additions and corrections resulted in not too many new errors and misprints.

Chapters 1 and 2 contain only 2–3 pages of new material like in sections 1.5, 2.5. Chapter 3 is considerably expanded. In particular we give more convenient definition of pseudo-convexity for second order equations and included boundary terms in Carleman estimates (Theorem 3.2.1$'$) and Counterexample 3.2.6. We give a new, shorter proof of Theorem 3.3.1 and new Theorems 3.3.7, 3.3.12, and Counterexample 3.3.9. We revised section 3.4, where a new short proof of exact observability inequality in given: proof of Theorem 3.4.1 and Theorems 3.4.3, 3.4.4, 3.4.8, 3.4.9 are new. Section 3.5 is new and it exposes recent progress on Carleman estimates, uniqueness and stability of the continuation for systems. In Chapter 4 we added to sections 4.5, 4.6 some new material on size evaluation of inclusions and on small inclusions. Chapter 5 contains new results on identification of an elliptic equation from many local boundary measurements (Theorem 5.2.2$'$, Lemma 5.3.8), a counterexample to stability, a brief description of recent complete results on uniqueness of conductivity in the plane case, some new results on identification of many coefficients and of quasilinear equations insectiosn 5.5, 5.6, and changes and most recent results on uniqueness for some important systems, like isotropic elasticity systems. In Chapter 7 we inform about new developments in boundary rigidity problem. Section 7.4 now exposes a complete solution of the uniqueness problem in the attenuated plane tomography over straight lines (Theorem 7.4.1) and an outline of relevant new methods and ideas. In section 8.2 we give a new general scheme of obtaining uniqueness results based on Carleman estimates and applicable to a wide class of partial differential equations and systems (Theorem 8.2.2) and describe recent progress on uniqueness problem for linear isotropic elasticity system. In Chapter 9 we expanded the exposition in section 9.1

to reflect increasing importance of the final overdetermination (Theorems 9.1.1, 9.1.2) . In section 9.2 we expose new stability estimate for the heat equation transform (Theorem 9.2.1' Lemma 9.2.2). New section 9.3 is dedicated to emerging financial applications: the inverse option pricing problem. We give more detailed proofs in section 9.5 (Lemma 9.5.5 and proof of Theorem 9.5.2). In Chapter 10 we added a brief description of a new efficient single layer algorithm for an imporatnt inverse problem in acoustics in section 10.2 and a new section 10.5 on so-called range tests for numerical solutions of overdermined inverse problems.

Many exercises have been solved by students, while most of the research problems await solutions. Chapter 7 of the final version of the manuscript have been read by Alexander Bukhgeim, who found several misprints and suggested many corrections. The author is grateful to him for attention and help. He also thanks the National Science Foundation for long-term support of his research, which stimulated his research and the writing of this revision.

Wichita, Kansas Victor Isakov

# Preface to the First Edition

This book describes the contemporary state of the theory and some numerical aspects of inverse problems in partial differential equations. The topic is of substantial and growing interest for many scientists and engineers, and accordingly to graduate students in these areas. Mathematically, these problems are relatively new and quite challenging due to the lack of conventional stability and to nonlinearity and nonconvexity. Applications include recovery of inclusions from anomalies of their gravitational fields; reconstruction of the interior of the human body from exterior electrical, ultrasonic, and magnetic measurements, recovery of interior structural parameters of detail of machines and of the underground from similar data (non-destructive evaluation); and locating flying or navigated objects from their acoustic or electromagnetic fields. Currently, there are hundreds of publications containing new and interesting results. A purpose of the book is to collect and present many of them in a readable and informative form. Rigorous proofs are presented whenever they are relatively short and can be demonstrated by quite general mathematical techniques. Also, we prefer to present results that from our point of view contain fresh and promising ideas. In some cases there is no complete mathematical theory, so we give only available results. We do not assume that a reader possesses an enormous mathematical technique. In fact, a moderate knowledge of partial differential equations, of the Fourier transform, and of basic functional analysis will suffice. However, some details of proofs need quite special and sophisticated methods, but we hope that even without completely understanding these details a reader will find considerable useful and stimulating material. Moreover, we start many chapters with general information about the direct problem, where we collect, in the form of theorems, known (but not simple and not always easy to find) results that are needed in the treatment of inverse problems. We hope that this book (or at least most of it) can be used as a graduate text. Not only do we present recent achievements, but we formulate basic inverse problems, discuss regularization, give a short review of uniqueness in the Cauchy problem, and include several exercises that sometimes substantially complement the book. All of them can be solved by using some modification of the presented methods.

Parts of the book in a preliminary form have been presented as graduate courses at the Johannes-Kepler University of Linz, at the University of Kyoto, and at Wichita State University. Many exercises have been solved by students, while most of the research problems await solutions. Parts of the final version of the manuscript have been read by Ilya Bushuyev, Alan Elcrat, Matthias Eller, and Peter Kuchment, who found several misprints and suggested many corrections. The author is grateful to these colleagues for their attention and help. He also thanks the National Science Foundation for long-term support of his research, which stimulated the writing of this book.

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# 1

# Inverse Problems

In this chapter we formulate basic inverse problems and indicate their applications. The choice of these problems is not random. We think that it represents their interconnections and some hierarchy.

An inverse problem assumes a direct problem that is a well-posed problem of mathematical physics. In other words, if we know completely a "physical device," we have a classical mathematical description of this device including uniqueness, stability, and existence of a solution of the corresponding mathematical problem. But if one of the (functional) parameters describing this device is to be found (from additional boundary/experimental) data, then we arrive at an inverse problem.

## 1.1 The inverse problem of gravimetry

The gravitational field $u$, which can be measured and perceived by the gravitational force $\nabla u$ and which is generated by the mass distribution $f$, is a solution to the Poisson equation

$$-\Delta u = f \tag{1.1.1}$$

in $\mathbb{R}^3$, where $\lim u(x) = 0$ as $|x|$ goes to $+\infty$. For modeling and for computational reasons, it is useful to consider as well the plane case ($\mathbb{R}^2$). Then the behavior at infinity must be $u(x) = Cln|x| + u_0(x)$, where $u_0$ goes to zero at infinity. One assumes that $f$ is zero outside a bounded domain $\Omega$, which is a ball or a body close to a ball (earth) in gravimetry. The direct problem of gravimetry is to find $u$ given $f$. This is a well-posed problem: Its solution exists for any integrable $f$, and even for any distribution that is zero outside $\Omega$; it is unique and stable with respect to standard functional spaces. As a result, the boundary value problem (1.1.1) can be solved numerically by using difference schemes, although these computations are not very easy in the three-dimensional case. This solution is given by the Newtonian potential

$$u(x) = \int_\Omega k(x-y)f(y)dy, \qquad k(x) = 1/(4\pi|x|) \tag{1.1.2}$$

(or $k(x) = -1/(2\pi)\ln|x|$ in $\mathbb{R}^2$). Practically we perceive and can measure only the gravitational force $\nabla u$.

The *inverse problem of gravimetry* is to find $f$ given $\nabla u$ on $\Gamma$, which is a part of the boundary $\partial\Omega$ of $\Omega$.

This problem was actually formulated by Laplace, but the first (and simplest) results were obtained only by Stokes in the 1860s and Herglotz about 1910 [Her]. We will analyze this problem in Sections 2.1–2.2 and 4.1. There is an advanced mathematical theory of this problem presented in a book of the author [Is4]. It is fundamental in geophysics, since it simulates recovery of the interior of the earth from boundary measurements of the gravitational field. Unfortunately, there is a strong nonuniqueness of $f$ for a given gravitational potential outside $\Omega$. However, if we look for a more special type of $f$ (like harmonic functions, functions independent of one variable, or characteristic functions $\chi(D)$ of unknown domains $D$ inside $\Omega$), then there is uniqueness, and $f$ can be recovered from $u$ given outside $\Omega$, theoretically and numerically. In particular, one can show uniqueness of $f = \chi(D)$ when $D$ is either star-shaped with respect to its center of gravity or convex with respect to one of the coordinates.

An important feature of the inverse problem of gravimetry is its ill-posedness, which creates many mathematical difficulties (absence of existence theorems due to the fact that ranges of operators of this problem are not closed in classical functional spaces) and numerical difficulties (stability under constraints is (logarithmically) weak, and therefore convergence of iterative algorithms is very slow, so numerical errors accumulate and do not allow good resolution). In fact, it was Tikhonov who in 1944 observed that introduction of constraints can restore some stability to this problem, and this observation was one of starting points of the contemporary theory of ill-posed problems.

This problem is fundamental in recovering the density of the earth by interpreting results of measurements of the gravitation al field (gravitational anomalies). Another interesting application is in gravitational navigation. One can measure the gravitational field (from satellites) with quite high precision, then possibly find the function $f$ that produces this field, and use these results to navigate aircrafts. To navigate aircraft one needs to know $u$ near the surface of the earth $\Omega$, and finding $f$ supported in $\bar{\Omega}$ gives $u$ everywhere outside of $\Omega$ by solving a much easier direct problem of gravimetry. The advantage of this method is that the gravitational field is the most stationary and stable of all known physical fields, so it is most suitable for navigation. The inverse problem here is used to record and store information about the gravitational field. This problem is quite unstable, but still manegeable. We discuss this problem in Sections 2.2, 2.3, 3.3, 4.1, and in Chapter 10.

Inverse gravimetry is a classical example of an inverse source problem, where one is looking for the right side of a differential equation (or a system of equations) from extra boundary data. Let us consider a simple example: in the second-order ordinary differential equation $-u'' = f$ on $\Omega = (-1, 1)$ in $\mathbb{R}$. Let $u_0 = u(-1)$, $u_1 = u'(-1)$; then

$$u(x) = u_0 + u_1(x+1) - \int_{-1}^{x} (x-y) f(y)\,dy \text{ when } -1 < x < 1.$$

Prescribing the Cauchy data $u, u'$ at $t = 1$ is equivalent to the prescription of two integrals

$$\int_\Omega (1-y)f(y)dy \text{ and } \int_\Omega f(y)dy.$$

We cannot determine more given the Cauchy data at $t = -1, 1$, no matter what is the original Cauchy data. The same information about $f$ is obtained if we prescribe any $u$ on $\partial\Omega$ and if in addition we know $u'$ on $\partial\Omega$. In particular, nonuniqueness is substantial: one cannot find a function from two numbers. If we add to $f$ any function $f_0$ such that

$$\int_\Omega v(y)f_0(y)dy = 0$$

for any linear function $v$ (i.e., for any solution of the adjoint equation $-v'' = 0$), then according to the above formulae we will not change the Cauchy data on $\partial\Omega$. The situation with partial differential equations is quite similar, although more complicated.

If $\nabla u$ is given on $\Gamma$, then $u$ can be found uniquely outside $\Omega$ by uniqueness in the Cauchy problem for harmonic functions using the assumptions on the behavior at infinity. Observe that given $u$ on $\partial\Omega \subset \mathbb{R}^3$ one can solve the exterior Dirichlet problem for $u$ outside $\Omega$ and find $\partial_\nu u$ on $\partial\Omega \in Lip$, so in fact we are given the Cauchy data there.

**Exercise 1.1.1.** Assume that $\Omega$ is the unit disk $\{|x| < 1\}$ in $\mathbb{R}^2$.

Show that a solution $f \in L_\infty(\Omega)$ of the inverse gravimetry problem that satisfies one of the following three conditions is unique. (1) It does not depend on $r = |x|$. (2) It satisfies the second-order equation $\partial_2^2 f = 0$. (3) It satisfies the Laplace equation $\Delta f = 0$ in $\Omega$.

In fact, in the cases (2) and (3), $\Omega$ can be any bounded domain with $\partial\Omega \in C^3$ with connected $\mathbb{R}^2 \backslash \bar{\Omega}$. {*Hint*: to handle case (1) consider $v = r\partial_r u - 2u$ and observe that $v$ is harmonic in $\Omega$. Determine $v$ in $\Omega$ by solving the Dirichlet problem and then find $f$. In cases (2) and (3) introduce new unknown (harmonic in $\Omega$) functions $v = \partial_2^2 u$ and $v = \Delta u$.}

**Exercise 1.1.2.** In the situation of Exercise 1.1.1 prove that a density $f(r)$ creates zero exterior potential if and only if

$$\int_0^1 rf(r)dr = 0.$$

{*Hint*: make use of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and of the expression for the Laplacian in polar coordinates,

$$\Delta = r^{-1}(\partial_r(r\partial_r) + \partial_\theta(r^{-1}\partial_\theta)).$$

Observe that for such $f$ the potential $u$ does not depend on $\theta$, and perform an analysis similar to that given above for the simplest differential equation of second order.}

What we discussed briefly above can be called the density problem. It is linear with respect to $f$. The domain problem when one is looking for the unknown $D$ is apparently nonlinear and seems (and indeed is) more difficult. In this introduction we simply illustrate it by recalling that the Newtonian potential $U$ of the ball $D = B(a; R) \subset \mathbb{R}^3$ of constant density $\rho$ is given by the formulae

$$(1.1.3) \quad U(x; \rho\chi(B(a; R))) = \begin{bmatrix} R^3\rho/3|x-a|^{-1} & \text{when } |x-a| \geq R; \\ R^2\rho/2 - \rho/6|x-a|^2 & \text{when } |x-a| < R. \end{bmatrix}$$

These formulae imply that a ball and its constant density cannot be simultaneously determined by their exterior potential ($|x-a| > R$). One can only find $R^3\rho$. Moreover, according to (1.1.2) and (1.1.3), the exterior Newtonian potential of the annulus $A(a; R_1, R_2) = B(a; R_2)\backslash \bar{B}(a; R_1)$ is $(R_2^3 - R_1^3)\rho/3|x-a|^2$, so only $\rho(R_2^3 - R_1^3)$ can be found. In fact, in this example the cavity of an annulus further deteriorates uniqueness. The formulae (1.1.3) can be obtained by observing the rotational (around $a$) invariancy of the equation (1.1.1) when $f = \rho\chi(B(a; R))$ and using this equation in polar coordinates together with the continuity of the potential and first order derivatives of the potential at $\partial D$.

We will give more detail on interesting and not completely resolved inverse problem of gravimetry in Section 4.1, observing that starting from the pioneering work of P. Novikov [No], uniqueness and stability results have been obtained by Prilepko [Pr], [PrOV], Sretensky, Tikhonov, and the author [Is4].

There is another interesting problem of potential theory in geophysics, that of finding the shape of the geoid $D$ given the gravitational potential at its surface. Mathematically, like the domain problem in gravimetry, it is a free boundary problem that consists in finding a bounded domain $D$ and a function $u$ satisfying the conditions

$$\Delta u = \rho \text{ in } D \subset \mathbb{R}^3, \qquad \Delta u = 0 \text{ outside } \bar{D},$$
$$u, \nabla u \in C(\mathbb{R}^3), \quad \lim u(x) = 0 \text{ as } |x| \to \infty,$$
$$u = g_0 \text{ on } \partial D,$$

where $g_0$ is a given function. To specify the boundary condition, we assume that $D$ is star-shaped, so it is given in polar coordinates $(r, \sigma)$ by the equation $r < d(\sigma)$, $|\sigma| = 1$. Then the boundary condition should be understood as $u(d(\sigma)\sigma) = g_0(\sigma)$, where $g_0$ is a given function on the unit sphere. This problem is called the Molodensky problem, and it was the subject of recent intensive study by both mathematicians and geophysicists. Again, despite certain progress, there are many challenging questions, in particular, the global uniqueness of a solution is not known.

To describe electrical and magnetic phenomena one makes use of single- and double-layer potentials

$$U^{(1)}(x; gd\Gamma) = \int_\Gamma K(x, y)g(y)d\Gamma(y)$$

and

$$U^{(2)}(x; gd\Gamma) = \int_\Gamma \partial_{\nu(y)} K(x, y) g(y) d\Gamma(y)$$

distributed with (measurable and bounded) density $g$ over a piecewise-Lipschitz bounded surface $\Gamma$ in $\mathbb{R}^3$. As in inverse gravimetry, one is looking for $g$ and $\Gamma$ (or for one of them) given one of these potentials outside a reference domain $\Omega$. The inverse problem for the single-layer potential can be used, for example, in gravitational navigation: it is probably more efficient to look for a single layer distribution $g$ instead of the volume distribution $f$. As a good example of a practically important problem about double layer potentials we mention that of exploring the human brain to find active parts of its surface $\Gamma_c$ (cortical surface). The area of active parts occupy not more than 0.1 of area of $\Gamma_c$. They produce a magnetic field that can be described as the double-layer potential distributed over $\Gamma_c$ with density $g(y)$, and one can (quite precisely) measure this field outside the head $\Omega$ of the patient. We have the integral equation of the first kind

$$G(x) = \int_{\Gamma_c} \partial_{\nu(y)} K(x, y) g(y) d\Gamma(y), \quad x \in \partial\Omega,$$

where $\Gamma_c$ is a given $C^1$-surface, $\bar{\Gamma}_c \subset \Omega$, and $g \in L_\infty(\Gamma_c)$ is an unknown function. In addition to its obvious ill-posedness, an intrinsic feature of this problem is the complicated shape of $\Gamma_c$. There have been only preliminary attempts to solve it numerically. No doubt a rigorous mathematical analysis of the problem (asymptotic formulae for the double-layer potential when $\Gamma_c$ is replaced by a closed smooth surface or, say, use of homogenization) could help a lot.

In fact, it is not very difficult to prove uniqueness of $g$ (up to a constant) with the given exterior potential of the double layer.

We observe that in inverse source problems one is looking for a function $f$ entering the partial differential equation $-\Delta u = f$ when its solution $u$ is known outside $\Omega$. If one allows $f$ to be a measure or a distribution of first order, then the inverse problems about the density $g$ of a single or double layer can be considered as an inverse source problem with $f = d\Gamma$ or $f = g\partial_\nu(d\Gamma)$.

## 1.2 The inverse conductivity problem

The conductivity equation for electric (voltage) potential $u$ is

(1.2.1) $$\operatorname{div}(a\nabla u) = 0 \text{ in } \Omega.$$

For a unique determination of $u$ one can prescribe at the boundary the Dirichlet data

(1.2.2) $$u = g_0 \text{ on } \partial\Omega.$$

Here we assume that $a$ is a scalar function, $0 < \varepsilon_0 \leq a$, that is measurable and bounded. In this case one can show that there is a unique solution $u \in H_{(1)}(\Omega)$ to the direct problem (1.2.1)–(1.2.2), provided that $g_0 \in H_{(1/2)}(\partial\Omega)$ and $\partial\Omega$ is Lipschitz. Moreover, there is stability of $u$ with respect to $g_0$ in the norms of these spaces. In other words, we have the well-posed direct problem.

Often we can assume that $a$ is constant near $\partial\Omega$. Then, if $g_0 \in C^2(\partial\Omega)$, the solution $u \in C^1$ near $\partial\Omega$, so the following classical Neumann data are well-defined:

$$a\partial_\nu u = g_1 \quad \text{on } \Gamma, \tag{1.2.3}$$

where $\Gamma$ is a part of $\partial\Omega \in C^2$. In general case, $\partial_\nu \in H_{(-1/2)}(\partial\Omega)$, so the data (1.2.3) are still well-defined.

The *inverse conductivity problem* is to find $a$ given $g_1$ for one $g_0$ (one boundary measurement) or for all $g_0$ (many boundary measurements).

In many applied situations it is $g_1$ that is prescribed on $\partial\Omega$ and $g_1$ that is measured on $\Gamma$. This makes some difference (not significant theoretically and computationally) in the case of single boundary measurements but makes almost no difference in the case of many boundary measurements when $\Gamma = \partial\Omega$, since actually it is the set of Cauchy data $\{g_0, g_1\}$ that is given. The study of this problem was initiated by Langer [La] as early as in the 1930s.

The inverse conductivity problem looks more difficult than the inverse gravimetric one: it is "more nonlinear." On the other hand, since $u$ is the factor of $a$ in the equation (1.2.1), one can anticipate that many boundary measurements provide much more information about $a$ than one boundary measurement. We will show later that this is true when the dimension $n \geq 2$. When $n = 1$, the amount of information about $a$ from one or many boundary measurements is almost the same.

This problem serves as a mathematical foundation to electrical impedance tomography, which is a new and promising method of prospecting the interior of the human body by surface electromagnetic measurements. On the surface one prescribes current sources (like electrodes) and measures voltage (or vice versa) for some or all positions of those sources. The same mathematical model works in a variety of applications, such as magnetometric methods in geophysics, mine and rock detection, and the search for underground water.

In the following exercise it is advisable to use polar coordinates $(r, \theta)$ in the plane and separation of variables.

**Exercise 1.2.1.** Consider the inverse conductivity problem for $\Omega = \{r < 1\}$ in $\mathbb{R}^2$ with many boundary measurements when $a(x) = a(r)$. Show that this problem is equivalent to the determination of $a$ from the sequence of the Neumann data $w'_k(1)$ of the solutions to the ordinary differential equations $-r(arw')' - k^2aw = 0$ on $(0, 1)$ bounded at $r = 0$ and satisfying the boundary condition $w(1) = 1$.

We will conclude this section with a discussion of the origins of equation (1.2.1), which we hope will illuminate possible applications of the inverse conductivity problem.

The first source is in Maxwell's system for electromagnetic waves of frequency $\omega$:

$$\operatorname{curl} \mathbf{E} = -i\omega\mu\mathbf{H}, \tag{1.2.4}$$
$$\operatorname{curl} \mathbf{H} = \sigma\mathbf{E} + i\omega\epsilon\mathbf{E},$$

where $\mathbf{E}$, $\mathbf{H}$ are electric and magnetic vectors and $\sigma$, $\epsilon$, and $\mu$ are respectively conductivity, electric permittivity, and magnetic permeability of the medium. In the human body $\mu$ is small, so we neglect it and conclude that $curl\, \mathbf{E} = 0$ in $\Omega$. Assuming that this domain is simply connected, we can claim that $\mathbf{E}$ is a potential field; i.e., $\mathbf{E} = \nabla u$. Since it is always true that div $curl\, \mathbf{H} = 0$, from second equation (1.2.4) we obtain for $u$ equation (1.2.1) with

$$a = \sigma + i\omega\epsilon. \tag{1.2.5}$$

Observe that in medical applications $\sigma$ and $\epsilon$ are positive functions of $x$ and $\omega$. In certain important situations one can assume that $\epsilon$ is small and therefore obtain equation (1.2.1) with the real-valued coefficient $a = \sigma$, which is to be found from exterior boundary measurements. This explains what the problem has to do with inverse conductivity. An important feature of the human body is that conductivites of various regions occupied by basic components are known constants, and actually one is looking for the shapes of these regions. For example, conductivities of muscles, lungs, bones, and blood are respectively 8.0, 1.0, 0.06, and 6.7.

In geophysics the same equation is used to describe prospecting by use of magnetic fields. Moreover, it is a steady-state equation for the temperature $u$. Indeed, if at the boundary of a domain $\Omega$ we maintain time-independent temperature $g(x)$, $x \in \partial\Omega$, then (Section 9.0) a solution of the heat equation $\partial_t U = \operatorname{div}(a\nabla U)$ in $\Omega$, $0 < t$, is (exponentially) rapidly convergent to a steady-state solution $u$ to the equation (1.2.1) with the Dirichlet boundary condition (1.2.2). The function $a$ then is called the thermal conductivity of the medium and is to be found in several engineering applications.

So, the inverse conductivity problem applies to a variety of situations when important interior characteristics of a physical body are to be found from boundary experiments and observations of fundamental fields.

## 1.3 Inverse scattering

In inverse scattering one is looking for an object (an obstacle $D$ or a medium parameter) from results of observations of so-called field generated by (plane) incident waves of frequency $k$. The field itself (acoustic, electromagnetic, or elastic) in the simplest situation of scattering by an obstacle $D$ is a solution $u$ to the Helmholtz equation

$$-\Delta u - k^2 u = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{D} \tag{1.3.1}$$

satisfying the homogeneous Dirichlet boundary condition

(1.3.2) $$u = 0 \text{ on } \partial D \quad \text{(soft obstacle)}$$

or another boundary condition, like the Neumann condition

($1.3.2_h$) $$\partial_\nu u + bu = 0 \text{ on } \partial D \text{ (hard obstacle).}$$

This solution is assumed to be the sum of the plane incident wave $u^i$ and a scattered wave $u^s$ that is due to the presence of an obstacle

(1.3.3) $$u(x) = u^i(x) + u^s(x),$$

where $u^i(x) = \exp(ik\xi \cdot x)$. In most cases in scattering theory one assumes that $\mathbb{R}^3 \backslash \overline{D}$ is connected, and $D$ is a bounded open set with Lipschitz boundary. In some situations spherical incident waves $u^i$ (depending only on $|x - a|$) are more useful and natural.

Basic examples of scattering by a medium are obtained when one replaces equation (1.3.1) by the equation

(1.3.4) $$-\Delta u + (c - ikb_0 - k^2 a_0)u = 0 \quad \text{in } \mathbb{R}^3.$$

The coefficients $a_0, b_0, c$ are assumed to be in $L_\infty(\Omega)$ for a bounded domain $\Omega$, with $b_0, c$ zero outside $\Omega$ and $a_0 > \varepsilon > 0$ and equal to 1 outside $\Omega$. In the representation (1.3.3) the first term in the right side is a simplest solution of the Helmholtz equation in $\mathbb{R}^3$ when there is no obstacle or perturbation of coefficients. In the presence of obstacles $u$ is different from $u^i$, and the additional term $u^s$ can be interpreted as a wave scattered from an obstacle or perturbation.

It can be shown that for any incident direction $\xi \in \Sigma$ there is a unique solution $u$ of the scattering problem (1.3.1), (1.3.2), (1.3.3) or (1.3.3), (1.3.4), where the scattered field satisfies the Sommerfeld radiation condition

(1.3.5) $$\lim r(\partial_r u^s - iku^s) = 0 \text{ as } r \text{ goes to } +\infty.$$

This condition guarantees that the wave $u(x)$ is outgoing. For soft obstacles if we assume $\partial D \in C^2$, then $u \in C^1(\mathbb{R}^3 \backslash D)$. For scattering by medium we have $u \in C^1$ under the given assumptions on $c, b_0, a_0$ and $u \in C^2$ when $c, b_0, a_0 \in C^1$. We discuss solvability in more detail in Chapter 6.

Any solution to the Helmholtz equation outside of $\Omega$ that satisfies condition (1.3.5) admits the representation

(1.3.6) $$u^s(x) = \exp(ikr)/r\mathcal{A}(\sigma, \xi; k) + O(r^{-2}),$$

where $\mathcal{A}$ is called the *scattering amplitude*, or *far field pattern*.

The *inverse scattering problem* is to find a scatterer (obstacle or medium) from far field pattern.

This problem is fundamental mathematical model of exploring bodies by acoustic or electromagnetic waves. The inverse medium problem with $a_0 = 1$, $b_0 = 0$ is basic in quantum mechanics, as suggested by Schrödinger in the 1930s because quantum mechanical systems are not accessable by direct experiments, which can destroy them. Only far field pattern can be observed, and from this information one has to recover the potential $c$ of atomic interaction.

In fact, it is difficult to implement measurement of a complex-valued function $\mathcal{A}$ due to oscillations of its argument, so one has to recover a scatterer from $|\mathcal{A}|$. The restricted problem is even more difficult due to partial loss of information and additional nonlinearity. Later on we will assume that $\mathcal{A}$ is given.

Even at this early stage we can introduce the so-called Lippman-Schwinger integral equation

$$(1.3.7) \quad u(x,\xi;k) = e^{ik\xi\cdot x} - \int_\Omega e^{ik|x-y|}/(4\pi|x-y|)c(y)u(y,\xi;k)dy,$$

which is equivalent to the differential equation (1.3.4) (with $a_0 = 1$ and $b_0 = 0$) and to the radiation condition (1.3.6) for the scattered wave due to the easily verifiable properties of the radiating fundamental solution $e^{ik|x-y|}/(4\pi|x-y|)$. Writing $|x-y| = |x|(1-|x|^{-2}x\cdot y + O(|x|^{-2}))$ where $O$ is uniform with respect to $y \in \Omega$ and comparing (1.3.7) and (1.3.6), we obtain the well-known representation for the scattering amplitude.

$$(1.3.8) \qquad \mathcal{A}(\sigma,\xi;k) = -1/(4\pi)\int_\Omega e^{-ik\sigma\cdot y}c(y)u(y,\xi;k)dy.$$

By using basic Fourier analysis one can show that the second term in (1.3.7) is uniformly (with respect to $x \in \Omega$) convergent to zero when $k \to +\infty$, so the scattering solution $u(y,\xi;k)$ behaves like $e^{ik\xi\cdot y}$. These facts easily lead to the uniqueness of $c$ when $\mathcal{A}$ is given for all values of $\sigma, \xi \in S^2$ and $k \in \mathbb{R}$. Indeed, let us pick up any $\eta \in \mathbb{R}^3$ and let $k \to +\infty$ keeping $k(\xi - \sigma) = \eta$. Then the limit of the right side of (1.3.8) is the Fourier transformation $\hat{c}(\eta)$ of $c$, which uniquely determines $c$.

A similar approach was found by J. Keller in 1958 to show that the high-frequency behavior of the scattering amplitide of a soft, strictly convex obstacle $D$ uniquely (and in a stable way) determines $D$. His crucial observation was that the first term of $\mathcal{A}(\sigma, -\sigma; k)$ for large $k$ determines the Gaussian curvature $\mathcal{K}(y(\sigma))$ of $\partial D$ at its point $y(\sigma)$ where a plane $y\cdot\sigma = s$ (with smaller $s$) intersects $\partial D$ at a point. The next step is a solution of the Minkowski problem of reconstruction of convex $D$ from its its Gaussian curvature, which was well understood at that time. Unfortunately, the high-frequency approach has serious drawbacks from a practical point of view because in many cases scattered fields decay quite rapidly due to damping when frequency is growing.

We will try to give some explanation of the origins of the inverse scattering problem. Our first example is the attempt to recover shapes of obstacles or important parameters of an acoustically oscillating medium from observation at large distances. The acoustic system linearized around the steady state (velocity $v_0 = 0$, pressure $p = p_0$, density $\rho = \rho_0(x)$) can be written as the so-called acoustic equation

$$a_0^2\partial_t^2 U - \rho_0 \operatorname{div}(\rho_0^{-1}\nabla U) + b_0\partial_t U = 0$$

for the linear term $U$ of the small perturbation of $p = p_0 + U + \cdots$ around the steady state. Here $a_0(x)$ is the inverse to the speed of sound and $b_0(x)$ is the damping/attenuation coefficient. The standard assumption in acoustics is that $\nabla \rho_0$ is small relative to $\rho_0$, so one lets $\rho_0 = 1$. When we consider time-harmonic oscillations $U(x,t) = u(x)e^{-ikt}$, we will have for the time harmonic waves $u$ the partial differential equation (1.3.4) with $c = 0$. In acoustics the waves of high frequency decay very rapidly due to the damping factor $b_0$, so practically, one can receive waves only with $k_* < k < k^*$ (mid frequencies). In air $k_* = 0.1$ and $k^* = 30$, while in water these numbers are 0.1 and 10 (distances in meters).

In electromagnetic prospecting one starts with the time-dependent Maxwell's equations to arrive at equations (1.2.4) of time-harmonic oscillations of frequency $\omega$. We will discuss this system and the inverse problems in more detail in section 5.8. One popular assumption is that outside of the reference medium the electromagnetic parameters $\mu, \epsilon, \sigma$ are constant. Right now we give values of these parameters in a typical solution. One of used frequencies is $\omega = 5,000$, and then for water, $\sigma/(\omega\epsilon) = 0.04$

In inverse scattering one is looking for a domain $D$ (whose boundary can be described by a function of two variables) or for functions $c, a_0, b_0$ of three variables given a function $\mathcal{A}(\sigma, \xi; k)$ of five variables. This is an overdetermined problem, so mathematically and from an applied point of view it is reasonable to consider partial scattering data. One can fix the frequency $k$ and incident direction $\xi$ by observing the results of scattering for all directions $\sigma$ of the receiver (this is appropriate for the obstacle problem). When one is looking for $c$ it makes sense to consider either fixed $k$ and all $\sigma$ and $\xi$, or to use all $\sigma = -\xi$ and $k$ (backscattering). In these restricted inverse problems it is much more difficult to prove uniqueness. At present there are certain uniqueness theorems and many challenging questions. It is interesting that the idea of high frequencies has been used by Sylvester and Uhlmann for $\mathbb{R}^3$ to prove uniqueness of potential $c$ with the data given at a fixed physical frequency. We will discuss inverse scattering in more detail in Chapter 6, and we refer to the books of Chadan and Sabatier [ChS], of Colton and Kress [CoKr], and of Lax and Phillips [LaxP2], to the encyclopedic collection [Sc], as well as to the paper of Faddeev [F].

## 1.4 Tomography and the inverse seismic problem.

The task of *integral geometry*, or (in applications) *tomography*, is to find a function $f$ given the integrals

$$\int_\gamma f\, d\gamma \tag{1.4.1}$$

over a family of manifolds $\{\gamma\}$. The case when the $\gamma$ are straight lines in $\mathbb{R}^2$ is quite important because it models X-rays. Then the integrals (1.4.1) are available from medical measurements. Uniqueness of recovery of $f$ and an explicit reconstruction formula were due to Radon in 1917, so often this problem is called after him.

But the applied importance of this problem has been made clear by Cormack and Hounsfield, who developed in the 1960s an effective numerical and medical technique for exploring the interior of the human body for diagnostic purposes. In 1979 they received the Nobel Prize for this work.

If a seismic (elastic) wave propagates in the earth, it travels along geodesics $\gamma(x, y)$ of the Riemannian metric $a^2(x)|dx|^2$. In simplest case, $a$ is the density of the earth. The travel time from $x$ to $y$ is then the integral

$$\tau(x, y) = \int_{\gamma(x,y)} d\gamma, \tag{1.4.2}$$

which is available from geophysical measurements.

The *inverse seismic problem* is to find $a$ given $\tau(x, y)$ for $x, y \in \Gamma$ that is a part of $\partial\Omega$.

Seismic waves can be artificially incited by some perturbations (microexplosions) on part $\Gamma$ of the surface of the Earth, and seismic measurements can be implemented with high precision. The spherically symmetric model of earth ($\Omega$ is a ball and $a$ depends only on the distance to its center) was considered by Herglotz in the 1910s, who developed one of the first mathematical models in geophysical prospecting.

We will consider the even simpler (but still interesting) case when $\Omega$ is the half-space $\{x_3 < 0\}$ in $\mathbb{R}^3$ and $a = a(x_3)$. Since $a$ does not depend on $x_2$, the curve $\gamma(x, y)$ will be contained in the plane $\{x_2 = 0\}$, provided that both $x$ and $y$ are in this plane. Later on we will drop the variable $x_2$. It is known (and not hard to show) that the function $\tau$ satisfies the following eikonal equation:

$$a^2(x_3)((\partial_1\tau)^2 + (\partial_3\tau)^2) - 1 = 0,$$

or

$$\partial_3\tau + \sqrt{a^{-2} - (\partial_1\tau)^2} = 0,$$

where $\partial_j$ is partial differentiation with respect to $x_j$, and $\sqrt{\phantom{x}}$ can be with $+$ or $-$ depending on the part of $\gamma$. It is clear that $\tau(x, y) = \tau(y, x)$, so later on we will fix $y = (0, 0)$ and consider travel time only as a function of $x$, which we will treat as an arrival point. The known theory of nonlinear partial differential equations of first order [CouH, p. 106] is based on the following system of ordinary differential equations for characteristics:

$$\frac{dx_1}{dx_3} = p_1(a^{-2} - p_1^2)^{-1/2}, \frac{dp_1}{dx_3} = 0,$$

where $p_1 = \partial_1\tau$. When $a$ is known, a solution $\tau$ to the eikonal equation in $\Omega$ is uniquely determined by the initial data on the line $\{x_3 = 0\}$, and according to the known theory of differential equations of first order it is formed from characteristics that are original geodesics. When $a$ is an increasing function of $x_3$ so that it goes to zero when $x_3$ goes $-\infty$, these characteristics consist of two symmetric parts, where $x_1$ is monotone with respect to $x_3$, which have a common point $(x_{1m}, x_{3m})$ with the minimum of $x_3$ over the geodesics achieved at $x_{3m}$. Integrating the first of our differential equations for characteristics over the interval $(0, x_{3m})$, we will obtain a half–travel time along the geodesics

$$\tau((2x_{1m}, 0), y) = 2\int_0^{x_{3m}} p_1(a^{-2}(s) - p_1^2)^{-1/2} ds = 2\int_\alpha^p t^{1/2}(p-t)^{-1/2} g'(t) dt$$

when we use the substitution of the inverse function $t = a^2(s)$ so that $s = g(t)$ and let $p = p_1^{-2}$, $\alpha = a^2(0)$. The upper limit is $p$ because at the point $(x_{1m}, x_{3m})$ the geodesic is parallel to the $x_3$-axis, and therefore the denominator in both integrals is zero. Now, $\alpha$ can be considered as a known function as well as $p$ as a function of $x_{1m}$ because these quantities are measured at $\{x_3 = 0\}$. So we arrive at the following integral equation:

$$\int_\alpha^p (p-t)^{\lambda-1} f(t) dt = F(p), \qquad \alpha < p < \beta, \tag{1.4.3}$$

$\lambda = 1/2$, with respect to $f(t)(= t^{1/2} g'(t))$, which is the well-known Abel integral equation. It arises also in other inverse problems (tomography (see Section 7.1) and determining the shape of a hill from travel times of a heavy ball up and down (see paper of J. Keller [Ke])). Equation (1.4.3) is one of the earliest inverse problems. It was formulated and solved by Abel around 1820.

**Exercise 1.4.1.** Show that the Abel equation (1.4.3) has the unique solution

$$f(t) = \frac{\sin \pi\lambda}{\pi} \frac{d}{dt} \int_\alpha^t (t-p)^{-\lambda} F(p) dp, \quad \alpha < t < \beta, \tag{1.4.4}$$

provided that $0 < \lambda < 1$ and $f \in C[\alpha, \beta]$ exists.

{*Hint*: Multiply both sides of (1.4.3) by $(s-p)^{-\lambda}$, integrate over the interval $(\alpha, s)$, change the order of integration in the double integral on the left side, and make use of the known identity

$$\int_0^1 \theta^{-\lambda}(1-\theta)^{\lambda-1} d\theta = \frac{\pi}{\sin \pi\lambda}$$

to calculate the interior integral with respect to $p$.}

More general equations of Abel type as well as their theory and applications can be found in the book of Gorenflo and Vessella [GorV].

It is interesting and important to consider the more general problem of finding $f$ and $a$ from the integrals

$$\int_{\gamma(x,y)} \rho(, \gamma) f \, d\gamma, \tag{1.4.5}$$

where $\rho$ is a partially unknown (weight) function that reflects diffusion (attenuation) in applied problems. Not much is known about this general problem. We will describe some results about this problem in Chapter 7.

The problems of integral geometry are closely related to inverse problems for the hyperbolic equation

$$a_0\partial_t^2 u + b_0\partial_t u - \mathrm{div}(a\nabla u) + cu = f \text{ in } \Omega \times (0, T) \tag{1.4.6}$$

with zero initial data

$$u = \partial_t u = 0 \quad \text{on } \Omega \times \{0\} \tag{1.4.7}$$

and the lateral Neumann boundary data

$$a\partial_\nu u = g_1 \text{ on } \partial\Omega \times (0, T). \tag{1.4.8}$$

The initial boundary value problem (1.4.6)–(1.4.8) has a unique solution $u$ for any (regular) boundary data, provided that $a, b_0, c, f$ are given and sufficiently smooth.

The inverse problem is to find $a, b_0, c, f$ (or some of them) from the additional boundary data

$$u = g_0 \text{ on } \gamma \times (0, T), \tag{1.4.9}$$

where $\gamma$ is a part of $\partial\Omega$. We will discuss this problem in Chapters 7 and 8. It is far from a complete solution in the case of one boundary measurement. But once the lateral Neumann-to-Dirichlet map $\Lambda_l : g_1 \to g_0$ is given, the problem was recently solved in several important cases. Under reasonable assumptions one can guarantee uniqueness and stability of recovery of $b_0, c$ when $T$ and $\Gamma$ are sufficiently large and $g_0$ is given for all smooth $g_1$. The situation with $a$ is more complicated: it can be uniquely determined only up to a conformal transformation of a corresponding Riemannian manifold, and the stability of a known hypothetical reconstruction is quite weak. In any case, if $a = 1$, $b_0 = c = 0$ there is a uniqueness theorem due to Belishev that is valid for any $\gamma$ and guarantees uniqueness of recovery of $a_0$ in the domain that can be reached by waves initiated and observed on $\gamma$. If time $T$ is large enough, this domain is the whole of $\Omega$.

In the isotropic case, behavior of elastic materials and elastic waves is governed by the elasticity system for the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$,

$$\rho\partial_t^2\mathbf{u} - \mathrm{div}(\mathcal{A}(\epsilon(\mathbf{u})) = \mathbf{f} \text{ in } \Omega \times (0, T), \tag{1.4.10}$$

where $\epsilon(\mathbf{u})$ is the stress tensor with the components $\frac{1}{2}(\partial_l u_m + \partial_m u_l)$ and $\mathcal{A}$ is the elastic tensor with the components $a_{jklm}(x)$. In the general case these components satisfy the symmetry conditions $a_{jklm} = a_{lmjk} = a_{kjlm}$, and in the important simplest case of classical elasticity,

$$a_{jklm} = \lambda\delta_{jk}\delta_{lm} + \mu(\delta_{jl}\delta_{jk} + \delta_{jm}\delta_{kl}).$$

The system (1.4.10) is considered together with the initial conditions that prescribe initial displacements and velocities and a lateral boundary condition, e.g., prescribing normal components of the stress tensor $\mathcal{A}\epsilon(\mathbf{u})$ on $\partial\Omega \times (0, T)$. As in electromagnetic scattering one can consider time-periodic elastic vibrations and elastic scattering problems. Only recently has there been some progress in understanding inverse problems in elasticity, and we report on certain results in

Sections 5.8 and 8.2. The contemporary state of the inverse seismic problem based on general linear system of (anisotropic) elasticity is described by de Hoop [I2] and de Hoop and Stolk [DS].

The inverse problems for hyperbolic equations and problems of integral geometry are closely related. One can show that the data of the inverse problem for the hyperbolic equation determine the data for tomographic and seismic problems. To do so one can use special high-frequency (beam) solutions or propagation of singularities of nonsmooth (in particular) fundamental solutions.

Sometimes tomographic approximation is not satisfactory for applications (in particular, it does not properly describe diffusion), while multidimensional inverse problems for hyperbolic equations are hard to solve. As a good compromise one can consider inverse problems for the transport equation.

$$\partial_t u + v \cdot \nabla u + b_0 u = \int_W K(\ , v, w) u(\ , w) dw + f \tag{1.4.11}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$, where $u(x, t, v)$ is the density of particles and $K(x, v, w)$ is the so-called collision kernel. Let $\partial\Omega_v$ be the "illuminated" part $\{x \in \partial\Omega : \nu(x) \cdot v < 0\}$. One can show that the initial boundary value problem (with data on $\partial\Omega_v$) for the nonstationary transport equation (1.4.11) has a stable unique solution (in appropriate natural functional spaces) under some reasonable assumptions. The inverse problem is to find the diffusion coefficient $b_0$, the collision kernel $K$, and the source term $f$ from $u$ given on $\partial\Omega$ for some or for all possible boundary data and zero initial conditions. Not much is known about the general problem, though there are some partial results. Quite important is a stationary problem when one drops $t$-dependence and the initial conditions. The inverse problem is more difficult, and even the simplest questions have no answers yet. We discuss these problems in Section 7.4. Observe that if $b_0 = 0$, $K = 0$, and $f$ is unknown, we arrive at tomography over straight lines, which is satisfactorily understood. But when $b_0 \geq 0$ is not zero there are many challenging open questions including the fundamental one about the uniqueness of $f$ and $b_0$ .

## 1.5 Inverse spectral problems

The domain problem was formulated already by Sir A. Shuster, who in 1882 introduced spectroscopy as a way "to find a shape of a bell by means of the sounds which it is capable of sending out." More rigorously, it has been posed by Bochner in the 1950s and then in the well-known lecture of Marc Kac, "Can one hear the shape of a drum?" in 1966. The mathematical question is as follows. Can a domain $D$ be determined by the eigenvalues $\lambda = \lambda_k$ of the Dirichlet problem

$$-\Delta u + \lambda u = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D. \tag{1.5.1}$$

Physically, eigenvalues correspond to resonance frequencies, so $\{\lambda_k\}$ can be considered as natural "exterior" information (there is no need to know $\partial D$ to prescribe the data). It is obvious that eigenvalues do not change under isometries (rotations, translations, and reflections), so only the shape of $D$ can be determined. Even before the paper of Kac was published, John Milnor found a counterexample (a torus in $\mathbb{R}^{17}$). Later on, Vigneras [Vi] obtained counterexamples of nonisometric $n$-dimensional compact manifolds with the same spectra, $n = 2, 3, \ldots$. Recently, Gordon, Webb, and Wolpert [GoWW] found two nonisometric isospectral polygons giving a negative answer to the problem of Kac for domains with piecewise smooth boundaries. At present it is unknown whether domains with smooth boundaries can be found from their resonances. It even unknown whether such domains are isolated (modulo an isometry). For the most recent results we refer to the paper of Osgood, Phillips, and Sarnak [OsPS].

After the famous paper of H. Weyl (1911) with a proof that $\lambda_k \sim c_n(k/V)^{2/n}$ for large $k$ ($c_n$ depends only on the dimension $n$ of the space, $V$ is volume of $D$), there was intensive study of asymptotic behavior of eigenvalues. In particular, the known Minakshisundaram-Pleijel expansion implies that $\sum \exp(-\lambda_k t)$ (the sum is over $k = 1, 2, \ldots$) when $t$ goes to 0 behaves like $V(4\pi t)^{-n/2} + C/6(4\pi t)^{-n/2}t +$ bounded terms (provided that $n = 2, 3$). Here $C$ is the integral of scalar curvature over $D$. So the eigenvalues uniquely determine $V$ and $C$, and therefore the Euler characteristic in the plane case. There are similar asymptotic formulae uniquely determining the length of $\partial D$ in the plane case. For recent developments we refer the reader to the papers of Guillemin [Gui] and of Guillemin and Melrose [GuM].

The problem is interesting also for other elliptic equations and systems (like the biharmonic equations and the elasticity system) and for different boundary conditions. It makes sense to collect eigenvalues for one $\Omega$ that correspond to different equations and boundary conditions and use them together for identification of $\Omega$.

A more general question is about the uniqueness of a compact Riemannian manifold with given eigenvalues of the associated Laplace operator.

We will not discuss inverse spectral problems in this book any further. We rather refer to the book of Berard [Ber] and the short review paper of Protter [Pro].

A related question is about reconstruction of the coefficient $c$ (potential) of the elliptic operator from eigenvalues $\lambda$ of the operator

$$(-\Delta + c - \lambda)u = 0 \qquad \text{in } \Omega \tag{1.5.2}$$

under the Dirichlet boundary conditions $u = 0$ on $\partial\Omega$ or other boundary conditions (e.g., Floquet periodicity conditions). In the one-dimensional case this problem is understood quite well due to work of Ambartzumijan, Borg, Gelfand and Levitan, Marchenko, and Krein. The answer for $\Omega = (0, l)$ is that eigenvalues of the Dirichlet problem uniquely determine an even with respect to $l/2$ potential $c$, and two sets of eigenvalues corresponding to Dirichlet and Neumann conditions uniquely determine any potential $c \in L_\infty(\Omega)$.

In fact, the inverse spectral problem is quite well understood for equation (1.5.2) in $\Omega = (0, l)$ with the general boundary conditions of Sturm-Liouville type

$$\cos\theta_0 u(0) + \sin\theta_0 u'(0) = 0, \cos\theta_l u(l) + \sin\theta_l u'(l) = 0.$$

The general second-order ordinary differential equation with sufficiently regular coefficients can be reduced to the one-dimensional equation (1.5.2) by using two known substitutions as has been done in Section 8.1. Since the Dirichlet eigenvalues do not determine the potential $c$ uniquely, one tries to use additional spectral information. The spectral data for this boundary value problem consist of eigenvalues $\lambda_1 < \cdots < \lambda_k < \ldots$ and in addition, the $L_2(0, l)$-norms of corresponding normalized eigenfunctions $u_k(; c)$, which can be found from eigenvalues of two different Sturm-Liouville-type problems. For the Neumann data the normalized eigenfunction $u_k(0; c) = 1$. One possible method of solution for the Neumann boundary data ($\theta_0 = \theta_l = \pi/2$) and for $l = \pi$ in case of $0 \le \lambda_k$ is to form the function

$$U(x, y) = \sum_{k=1}^{\infty}(\|u_k(; c)\|_2^{-1}(0, \pi) cos\sqrt{\lambda_k} x cos\sqrt{\lambda_k} y \\ - 2/\pi\, cos(k-1)x cos(k-1)y) + 1/\pi,$$

to solve the Volterra type integral equation (Gelfand-Levitan equation)

$$K(x, y) + \int_0^x K(x, s)U(s, y)ds + U(x, y) = 0, 0 < y < x,$$

and to find

$$c(x) = 2\frac{d}{dx}K(x, x).$$

For more general formulations and proofs we refer to the books of Marchenko [Mar] and Pöschel and Trubowitz [PoT].

When $n > 2$ the situation is much more complicated, and not very much is known. For the state of the art we refer to the papers of Eskin and Ralston [ER1] and of DeTurck and Gordon [DGI], [DGII].

Other additional data are values of the normal derivatives $\partial_\nu u_k(\ ; A)$ of orthonormal eigenfunctions $u_k$ of the more general elliptic boundary value problem

$$Au_k = \lambda_k a_0 u_k \text{ in } \Omega, \\ u_k = 0 \text{ on } \partial\Omega \tag{1.5.3}$$

on $\Gamma$, where $Au = -\operatorname{div}(a\nabla u) + cu$. Given these data, from Green's function

$$G(x, y; \lambda) = \sum_{k=1}^{\infty}(\lambda - \lambda_k)^{-1}u_k(x)u_k(y)$$

one uniquely determines the lateral Dirichlet-to-Neumann map for the elliptic equation with parameter $(A + \lambda a_0)u = 0$ in $\Omega$ and therefore for the corresponding hyperbolic equation $(a_0\partial_t^2 + A)u = 0$ on any cylinder $\Omega \times (0, T)$. Indeed, writing the solution of the Dirichlet problem for the elliptic equation with the parameter under the boundary condition $u = g$ on $\partial\Omega$ in terms of Green's function and taking the normal derivative, we obtain

$$\Lambda g = a\partial_\nu u = \int_{\partial\Omega}\sum_{k=1}^{\infty}(\lambda - \lambda_k)^{-1}\partial_\nu u(x)\partial_\nu u(y)g(y)d\Gamma(y).$$

This argument is applicable only to self-adjoint elliptic operators. Then one can benefit from inverse hyperbolic problems, at least in the one-dimensional case, where they are relatively simple and well understood. We give a complete treatment in Section 8.1. In the many-dimensional case substantial progress has been achieved by Belishev using methods of boundary control, and we report on it in Section 8.4.

The goal of this book is to describe recent results about uniqueness and stability of recovery of coefficients of partial differential equations from (overdetermined) boundary data. These problems are nonlinear, and most of them are not well posed in the sense of Hadamard. However, they represent the most popular mathematical model of recovery of unknown physical, geophysical, or medical objects from exterior observations and provide new, challenging mathematical questions that are attracting more and more researchers in many fields. In these problems uniqueness plays a very important role, since such a requirement implies that we have enough data to determine an object. Also, uniqueness implies stability under some natural constraints. The theory of stable numerical solutions of ill-posed problems using regularization (approximation by well-posed problems) was developed in the 1950–1960s by John and Tikhonov (for references see the books of Engl, Hanke, and Neubauer [EnHN], the author [Is4], Lavrentiev, Romanov, and Shishatskij [LaRS], Louis [Lou], and Tikhonov and Arsenin [TiA]). Stability is crucial for the convergence of solutions of regularized problems to the solutions sought. We focus on uniqueness and stability, and only in Chapter 10 we discuss new interesting numerical algorithms.

This book consists of ten chapters dealing with regularization of ill-posed problems, uniqueness and stability in the Cauchy problem, inverse problems for elliptic equations, scattering problems, and hyperbolic and parabolic equations. We formulate many results, and in many cases we give ideas or short outlines of proofs. In some important cases proofs are complete and sometimes new. In this way we are attempting to demonstrate methods that can be used in a variety of inverse problems. In addition, we give many exercises, ranging from illuminating and surprising examples to substantial additions to the main text; so not all of them are easy to solve. Besides, we formulate many unsolved problems that in our opinion are of importance for theory and applications.

Uniqueness and stability of the continuation of solutions of partial differential equations and systems plays a fundamental role in theory and applications of inverse problems, especially in case of single boundary measurements and obstacle problems. The key idea in this area was conceived by Carleman in 1938. Since his groundbreaking work on uniqueness in the Cauchy problem for first order systems in $\mathbb{R}^2$ with simple characteristics so called Carleman estimates are the basic tool used in hundreds of interesting papers. We review the contemporary state of this theory and give some new positive results and counterexamples.

Also in 1938 there was another new deep result due to P. Novikov who proved uniqueness of a star-shaped plane domain with given exterior gravitational potential. His uniqueness theorem still is one of best in inverse problems and his orthogonality method is widely used in inverse problems for elliptic and parabolic equations with maximum principle. We demonstrate the orthogonality method in different sections of the book.

More recent source of ideas is the fundamental paper of Sylvester and Uhlmann [SyU2] on the uniqueness of the Schrödinger equation with given Dirichlet-to-Neumann map in the three-dimensional case, where they resolved the question posed by Calderon [C] and generated hundreds of theoretical and applied publications on different aspects of the subject and on challenging and important problems concerning uniqueness and stability of a differential equation with given many boundary measurements. We discuss these problems for elliptic, parabolic, and hyperbolic equations and establish a connection with inverse scattering. The property of completeness of products of solutions of partial differential equations plays an important role in this theory, and we suggest quite a general scheme of proof of this property for equations with constant leading coefficients. Almost simulteneously with [SyU2] Belishev [Be3] suggested new powerful method of boundary control to demonstrate uniqueness in several important inverse hyperbolic problems with many lateral boundary data. His method combines some previous ideas from inverse spectral theory due to Gelfand, Levitan, M. Krein, and Marchenko with sharp uniqueness of the continuation results for hyperbolic equations of second order with time independent coefficients due to Tataru and based on new Carleman estimates. We expose some fragments of this theory as well as new developments in chapters 3 and 8.

Of obvious interest is the inverse conductivity problem with one boundary measurement, where there are preliminary theoretical results, but more questions than answers. That is why we start with this to some extent typical problem. Applications include not only electrical exploration, but magnetic, acoustic, and seismic exploration as well. This problem deals with the recovery of an unknown domain whose physical properties are quite different from those of the reference medium (different conductivity, permeability, density, etc.). Even linearizations of this inverse problem are highly nontrivial (say, the nonelliptic oblique derivative problem for the Laplace equation), and they appear to be quite challenging.

In Chapter 6 we collect many recent results about inverse scattering and try to reduce them to problems in finite domains. It is known that for any compact scatterer the scattering amplitude is an entire function, so operators that map scatterers into standard scattering data are highly smoothing operators, and the inverse to them must be quite unstable. Probably it is preferable to deal with the near field. We study stability of recovery of this field from the far field. Then we give a reduction of scattering problems to the Dirichlet-to-Neumann map and describe some recent results that cannot be obtained by this scheme. Besides, we discuss what inverse scattering can do for problems in finite domains (recall the $\bar{\partial}$-method, whose power was recently demonstrated by Astala, Nachman, and Päivärinta in their solution of the two-dimensional inverse conductivity problem).

Scattering is certainly related to hyperbolic equations, where integral geometry and uniqueness in the lateral Cauchy problem are also quite important. We discuss the interaction of all three topics in Chapters 3, 7, and 8. This Cauchy problem turns out to be quite stable as soon as the size of the surface with the Cauchy data and time are large enough; and when on the rest of the lateral boundary one prescribes a classical boundary value condition, the only remaining problem is with existence theorems. We feel that these stability estimates can be used in many problems, including identification of coefficients and integral geometry. Also, we should mention recent important results of Tataru [Tat2], who obtained an exact description of the uniqueness domain in the lateral Cauchy problem when the coefficients of a second-order hyperbolic equation are time-independent. In Chapter 7 we collect certain results of integral geometry, which are used in Chapter 8, and also discuss inverse problems for the transport equation. Chapter 8 is devoted to various inverse hyperbolic problems; in particular, we consider in some detail the one-dimensional case, the use of beam solutions, and methods of boundary control theory.

In Chapter 9 we are concerned with similar questions for parabolic equations. However, there is one specific problem (with final overdetermination) that we treat in more detail and that is in fact well-posed. This problem is crucial for the inverse option pricing problem which is discussed in section 9.3.

In Chapter 10 we collect promising and widely used numerical techniques, in particular relaxation and linear sampling methods.

The reader can find additional information in the books of Anger [Ang], Colton and Kress [CoK], Katchalov, Kurylev and Lassas [KKL], Sharafutdinov [Sh], recent conference proceedings [I1], [I2], [I3], [Ne] and in the review papers of Payne [P] (with an extensive bibliography up to 1975), and of Uhlmann [U1], [U2].

We use standard notation, but we recall some notation here for the convenience of the reader.

$B(x; R)$ is the ball of radius $R$ centered at a point $x$.

$S^{n-1}$ is the unit sphere $\partial B(0; 1)$ in the $n$-dimensional space $\mathbb{R}^n$.

$\chi(D)$ is the characteristic function of a set $D$ (1 on $D$, 0 outside $D$).

By *dist* between sets we understand the Hausdorff distance.

$\nu$ is the unit exterior normal to the boundary of a domain.

$meas_n$ stands for the $n$-dimensional Lebesgue measure.

$L_p(\Omega)$ is the space of functions with the finite norm $\|u\|_p(\Omega) = \left(\int_\Omega |u|^p\right)^{1/p}$, $1 \leq p$.

$H_{k,p}(\Omega)$ is the Sobolev space of functions on $\Omega$ (domain or $C^k$-smooth manifold) with partial derivatives of order $\leq k$ in $L_p(\Omega)$. The norm is denoted by $\| \ \|_{k,p}(\Omega)$. $H_{(k)} = H^2_k$. This space is defined also for negative $k$ by duality and for fractional $k$ by interpolation. We let $\| \ \|_{(k)} = \| \ \|_{k,2}$.

$C^k(\bar{\Omega})$ is the space of functions with continuous partial derivatives of order $\leq k$ in $\bar{\Omega}$, Hölder continuous of order $k - \{k\}$ when $k$ is fractional. The norm in this space is denoted by $| \ |_k(\Omega)$.

$\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ when $\alpha$ is a multiindex $(\alpha_1, \dots, \alpha_n)$, $\partial_j = \partial/\partial x_j$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

$A^*$ is the differential operator (formally) adjoint of $A$.

# 2

# Ill-Posed Problems and Regularization

In this chapter we consider the equation

(2.0) $$Ax = y$$

where $A$ is a (not necessarily linear) continuous operator acting from a subset $X$ of a Banach space into a subset $Y$ of another Banach space, and $x \in X$ is to be found given $y$. We discuss solvability of this equation when $A^{-1}$ does not exist by outlining basic results of the theory created in the 1960s by Ivanov, John, Lavrent'ev, and Tikhonov. In Section 2.1 we give definitions of well- and ill-posedness, together with important illustrational examples. In Section 2.2 we describe a class of equations (2.0) that can be numerically solved in a stable way. Section 2.3 is devoted to the variational construction of algorithms of solutions by minimizing Tikhonov stabilizing functionals. In Section 2.4 we show that stability estimates for equation (2.0) imply convergence rates for numerical algorithms and discuss the relation between convergence of these algorithms and the existence of a solution to (2.0). The final section, Section 2.5, describes some iterative regularization algorithms.

## 2.1 Well- and ill-posed problems

We say that equation (2.0) represents a well-posed problem in the sense of Hadamard if the operator $A$ has a continuous inverse from $Y$ onto $X$, where $X$ and $Y$ are open subsets of the classical spaces $C^k(\bar{\Omega})$, $H_{k,p}(\Omega)$, or their finite-codimensional subspaces. In other words, we require that

(2.1.1) for any $y \in Y$ there is no more than one $x \in X$ satisfying (2.0) (uniqueness of a solution);
(2.1.2) for any $y \in Y$ there exists a solution $x \in X$ (existence of a solution);
(2.1.3) $\|x - x^\bullet\|_X$ goes to 0 when $\|y - y^\bullet\|_Y$ goes to 0 (stability of a solution).

The condition that $X$ and $Y$ be subspaces of classical functional spaces is due to the fact that those spaces are quite natural for partial differential equations and mathematical physics. They reflect physical reality and serve as a basis for stable computational algorithms.

If one of the conditions (2.1.1)–(2.1.3) is not satisfied, the problem (2.0) is called ill-posed (in the sense of Hadamard).

We observe that these conditions are of quite different degrees of importance. If one cannot guarantee the uniqueness of a solution under any reasonable choice of $X$, then the problem does not make much sense and there is no hope of handling it. The condition (2.1.2) appears not as restrictive, because it shows only that we cannot describe conditions that guarantee existence. In fact, as shown later, even without this condition one can produce a stable numerical algorithm for finding $x$ given $Ax$. Moreover, in many important inverse problems mentioned in Chapter 1 it is not realistic to describe $\{Ax\}$. It looks as though without condition (2.1.3) problem (2.0) is not physical (as suggested by Hadamard in the 1920s) and is uncomputable, because practically, we never know exact data due to errors in measurement and computation. However, a reasonable use of convergence and a change of $X$ can fix this situation.

Now we consider examples of important and still not completely understood ill-posed problems.

EXAMPLE 2.1.1 (BACKWARD HEAT EQUATION). In the simplest case the problem is to find a function $u(x, t)$ satisfying the heat equation and the homogeneous lateral boundary conditions

$$\partial_t u - \partial_x^2 u = 0 \text{ in } \Omega \times (0, T), \qquad u = 0 \text{ on } \partial\Omega \times (0, T),$$

where $\Omega$ is the unit interval (0, 1), from the final data

$$u(x, T) = u_T(x), \qquad x \in (0, 1).$$

By using separation of variables we can see that the functions $u_k(x, t) = e^{-\pi^2 k^2 t} \sin(\pi k x)$ satisfy the heat equation and the boundary conditions. The initial data are $u_k(x, 0) = \sin(\pi k x)$. They have $C^0$-norm equal to 1 and $L_2$-norm $(1/2)^{\frac{1}{2}}$. The final data have $C^0$-norm $e^{-\pi^2 k^2 T}$ and $H_{(m)}$-norm $e^{-\pi^2 k^2 T}((1 + \cdots + (\pi k)^{2m})/2)^{1/2}$. If we define $Au_0 = u_T$, then the estimate $\|u_0\|_X \leq C\|u_T\|_Y$ is impossible when $X$, $Y$ are classical functional spaces: the norms of $u_{Tk}$ go to zero exponentially when the norms of the $u_{0k}$ are greater than 1/2. Therefore, the problem of finding the initial data from the final data is exponentially unstable in all classical functional spaces. This phenomenon is quite typical for many important inverse problems in partial differential equations.

The eigenfunctions $a_k(x) = 2^{1/2} \sin \pi k x$ of the operator $-\partial_x^2$ with eigenvalues $\pi^2 k^2$ form a complete orthonormal basis in the space $L_2(0, 1)$, so we can write

$$u(x, t) = \sum u_{0k} e^{-\pi^2 k^2 t} a_k(x),$$

where $u_{0k}$ is the Fourier coefficient of the initial data. In particular, we can see that the operator $A$ is continuous from $L_2(\Omega)$ into $L_2(\Omega)$. It is clear that existence of a solution with final data $u_T(x) = \sum u_{Tk} a_k(x)$ is equivalent to the very restrictive condition of the convergence of the series $\sum u_{Tk}^2 e^{\pi^2 k^2 T}$, which cannot be expressed in terms of the classical functional spaces defined via power growth of the Fourier coefficients $u_{Tk}$ (but not exponential!) with respect to $k$. A useful description of the range of the operator parabolic equations (see Section 3.1); so we have no existence theorem; and condition (2.1.2) is not satisfied.

In fact, conditions (2.1.)–(2.1.3) are not independent. For linear closed operators $A$ in Banach spaces, the conditions (2.1.1) and (2.1.2) imply condition (2.1.3) due to the Banach closed graph theorem, which implies that if a continuous linear operator maps a Banach space onto another Banach space and is one-to-one, then the inverse is continuous. Indeed, if $A$ maps an open subset $X$ of a subspace $X_1$ of a Banach space onto an open subset $Y$ of a subspace $Y_1$ (codim $X_1$ + codim $Y_1 < \infty$), then $A$ maps $X_1$ onto $Y_1$, both of which are Banach spaces. So by the Banach theorem the inverse $A^{-1}$ is continuous from $Y_1$ into $X_1$ with respect to the norms in $X$ and $Y$, and we have (2.1.3).

**Exercise 2.1.2 (A Nonhyperbolic Cauchy Problem for the Wave Equation).** Show that the Cauchy problem

$$\partial_t^2 u - \partial_1^2 u - \partial_2^2 u = 0 \text{ in } (0, T) \times \Omega, \quad u = g_0, \partial_2 u = g_1 \text{ on } (0, T) \times \Gamma,$$

where $\Omega = \{0 < x_1 < 1, 0 < x_2 < H\}$ and $\Gamma$ is the part $\{0 < x_1 < 1, 0 = x_3\}$ of its boundary, is ill-posed in the sense of Hadamard.

{*Hint*: Make use of separation of variables to construct a sequence of solutions that are bounded (with a finite number of derivatives) on $\Gamma$ while growing exponentially at a distance from $\Gamma$.}

In fact, there is exponential instability as in Example 2.1.1.

This problem was analyzed initially by Hadamard in his famous book [H], pp. 26, 33, 254–261, where there is an interesting description of the pairs $\{g_0, g_1\}$ that are the Cauchy data for some solution $u$.

EXAMPLE 2.1.3 (INTEGRAL EQUATIONS OF THE FIRST KIND). Consider equation (2.0) (with $x$ replaced by a function $f$ defined on $\Omega$ and $y$ replaced by a function $F$ defined on $\Omega_1$, $\Omega, \Omega_1 \subset \mathbb{R}^n$) when

$$Af(x) = \int_\Omega K(x, y) f(y) dy, \tag{2.1.4}$$

with the kernel $K$ continuous on $\bar{\Omega} \times \bar{\Omega}_1$. The operator $A$ is completely continuous from $L_2(\Omega)$ into $L_2(\Omega_1)$.

An important example of such equations is obtained with the Riesz kernels $K(x, y) = |x - y|^\beta$ when $\bar{\Omega}$ does not intersect $\bar{\Omega}_1$. When $n = 3$ and $\beta = -1$, we have the inverse problem of gravimetry, which is discussed in Section 4.1, and when $\beta = -2$, we will have the integral equation related to the linearized inverse conductivity problem (see Section 4.5) and to some inverse problems of scattering theory.

**Exercise 2.1.4.** Assume that $\Omega$ is the unit ball $|y| < 1$ in $\mathbb{R}^3$ and $\Omega_1$ is the annular domain $\{2 < |y| < 3\}$. Show that the integral equation $Af = F$ with the Riesz kernel represents an ill-posed problem.

{*Hint*: show that the operator $A$ maps the space $L_2(\Omega)$ into the space of functions that are real-analytic in some neighborhood of $\bar{\Omega}_1$ in $\mathbb{R}^3$ (and even in $\mathbb{C}^3$). This space is not a subspace of finite codimension of any of spaces $H_{k,p}(\Omega_1)$, so (2.1.2) is not satisfied. Actually, $A$ maps distributions supported in $\bar{\Omega}$ into analytic functions.}

Another important example is that of convolution equations. We let $\Omega = \Omega_1 = \mathbb{R}^n$, $X = Y = L_2(\Omega)$, and $K(x, y) = k(x - y)$. Then equation (2.0) takes the form

$$\int_{\mathbb{R}^n} K(x - y) f(y) dy = F(x), x \in \mathbb{R}^n. \tag{2.1.5}$$

To study and solve such equations one can use the Fourier (or Laplace) transform $f \to \hat{f}$, which transforms equation (2.1.5) into its multiplicative form

$$\hat{k}(\xi)\hat{f}(\xi) = \hat{F}(\xi).$$

**Exercise 2.1.5.** Show that this equation is ill-posed if and only if for any natural number $l$ the function $\hat{k}^{-1}(\xi)(1 + |\xi|)^{-l}$ is (essentially) unbounded on $\mathbb{R}^n$.

In particular, this equation is ill-posed for $k(x) = \exp(-|x|^2/(2T))$, which reflects the ill-posedness of the backward initial problem for the heat equation in the domain $\mathbb{R}^n \times (0, T)$. Indeed, it is known ([Hö2], sec.7.6) that $\hat{k}(\xi) = C\exp(-T|\xi|^2/2)$. Equation (2.1.5) or (2.0) then is equivalent to the well-known representation of the solution at a moment of time $T$ in terms of the initial data $u_0$.

## 2.2 Conditional correctness: Regularization

The equation (2.0) is called conditionally correct in a *correctness class* $X_M \subset X$ if it does satisfy the following conditions.

(2.2.1) A solution $x$ is unique in $X_M$; i.e., $x = x^\bullet$ as soon as $Ax = Ax^\bullet$ and $x$, $x^\bullet \in X_M$ (uniqueness of a solution in $X_M$).

(2.2.2) A solution $x \in X_M$ is stable on $X_M$; i.e., $\|x - x^\bullet\|_X$ goes to zero as soon as $\|Ax - Ax^\bullet\|_Y$ goes to zero and $x^\bullet \in X_M$ (conditional stability).

Sometimes we say also that a solution is unique and stable under a constraint, and $X_M$ is called a set of constraints.

We observe that the existence condition is completely eliminated. A reason is that in important applied problems it is almost never satisfied. Moreover, a stable numerical solution of the problem (2.0) can be obtained only under conditions (2.2.1) and (2.2.2), provided that a solution $x$ to equation (2.0) does exist. Certainly, a choice of the correctness class is crucial: it must not be so narrow as to reflect only some natural a priori information about a solution.

A function $\omega$ such that $\|x - x^\bullet\|_X \leq \omega(\|Ax - Ax^\bullet\|_Y)$ is called a *stability estimate*. In (2.2.2) this function may depend on a point $x$. In some cases it does not depend on $x \in X_M$, and then it is particularly interesting. We give stability estimates in Chapters 3–9 for some sets $X_M$ and for important inverse problems. A stability estimate must satisfy the condition $\lim \omega(\tau) = 0$ as $\tau$ goes to 0. It can and will be assumed monotone.

We make the simple but important observation that if $X_M$ is compact, then condition (2.2.1) implies condition (2.2.2) (uniqueness guarantees stability). Indeed, $A$ is continuous, by (2.2.1) it is one-to-one, and then the well-known topological lemma gives that $A^{-1}$ is continuous from $A(X_M)$ into $X$ with respect to the norms on $Y$ and $X$. Moreover, there is a stability estimate on $X_M$. This observation applied to the inverse problem of gravimetry by Tikhonov in 1943 was one of the ideas initiating the contemporary theory of stable solutions of ill-posed problems. Also, it emphasizes the mathematical role of uniqueness.

Let us consider the examples of Section 2.1. A solution of the backward heat equation is unique, so we can expect some stability. As shown in Section 3.1 (Exercise 3.1.2), there is a logarithmic stability estimate $\|u_0\| \leq \omega(\varepsilon) = -C\varepsilon_1 \ln(\varepsilon_1)$, where $\varepsilon_1 = -1/\ln\varepsilon$, $\varepsilon = \|u_T\|$, and $C$ depends on $M$, provided that

(2.2.3) $$\|u_0\|_2 + \|\partial_x^2 u_0\|_2 \leq M.$$

Here we let $X = Y = L_2(0, 1)$, and the operator $Au_0 = u_T$.

**Exercise 2.2.1.** Show that the set of functions $u_0$ satisfying condition (2.2.3) is compact in $X$.

We consider another operator $Au_\tau = u_T$ defined on solutions of the heat equations at the moment of time $t = \tau > 0$. Then we have a much better estimate $\|u_\tau\| \leq M\epsilon^{\tau/T}$ under the weaker constraint $\|u_0\| \leq M$, as shown in Section 3.1.

**Exercise 2.2.2.** Let $\Omega = (0, 1)$. Show that the set of functions $u(\tau)$, $\tau > 0$, where $u$ solves the heat equation in $\Omega \times (0, T)$, $\|u_0\|_0(\Omega) \leq M$ when $0 < t < T$, and $u$ is zero on $\partial\Omega \times (0, T)$, is compact in $X = L_2(0, 1)$.

The situation is more complicated if we consider Example 2.1.2. Then a solution $u$ is not unique in the domain $Q = \Omega \times (0, T)$ but only in a subdomain $Q_0$ described in Lemma 3.4.6 or in Exercise 3.4.7. The best-known stability estimate will be only of logarithmic type.

The basic idea in solving (2.0) is to use regularization, i.e., to replace this equation by a "close" equation involving a small parameter $\alpha$, so that the changed equation can be solved in a stable way and its solution is close to the solution of the original equation (2.0) when $\alpha$ is small.

In the following definition we need many-valued operators $R$ that map elements $y$ of $Y$ into subsets $\mathfrak{X}$ of $X$. We denote all closed subsets of $X$ by $\mathfrak{A}(X)$. The distance $d$ between two subsets $\mathfrak{X}$ and $\mathfrak{X}^\#$ is defined as $\sup_x \inf_v \|x - v\|_X + \sup_v \inf_x \|x - v\|_X$ where in the first term we take inf with respect to $v \in \mathfrak{X}^\#$ and then sup with respect to $x \in \mathfrak{X}$ and in the second term change $x$ and $v$. A many-valued operator $R$ is continuous at $y$ if $d(Ry^\bullet, Ry)$ goes to zero as $\|y^\bullet - y\|_Y$ goes to zero.

A family of continuous operators $R_\alpha$ from a neighborhood of $AX_M$ in $Y$ into $\mathfrak{A}(X)$ is called a *regularizer* to the equation (2.0) on $X_M$ when

$$\lim_{\alpha \to 0} R_\alpha Ax = x \quad \text{for any } x \in X_M. \tag{2.2.4}$$

The positive parameter $\alpha$ is called the *regularization parameter*.

Many-valued regularizers are necessary to treat nonlinear equations, while for linear $A$ we can normally build single-valued regularizers that are the usual continuous operators from $Y$ and $X$.

We observe that at least for linear $A$ that have no continuous inverse and for one-to-one $R_\alpha$ (examples are in Section 2.3), convergence in (2.2.4) is not uniform with respect to $x$ if $X_M$ contains an open subset of $X$. Indeed, assuming the contrary and using translations and scaling, we can obtain uniform convergence on $\{x : \|x\|_X = 1\}$. In particular, there is $\alpha$ such that $\|R_\alpha Ax - x\|_X \le 1/2\|x\|_X$. Let us consider the equation $x + (R_\alpha Ax - x) = R_\alpha y$. By the Banach contraction theorem it has a unique solution $x = By$. Moreover, by using the triangle inequality we obtain

$$\|x\|_X \le \|R_\alpha Ax - x\|_X + \|R_\alpha y\|_X \le \frac{1}{2}\|x\|_X + C\|y\|_Y$$

because $R_\alpha$ is continuous. Therefore, $\|By\|_X \le 2C\|y\|_Y$. We have $R_\alpha Ax = R_\alpha y$, and consequently $Ax = y$, so $A$ has the continuous inverse $B$, which is a contradiction. This shows that convergence in (2.2.4) is generally only pointwise, i.e., at any fixed $x$.

**Exercise 2.2.3.** Let $x \in X_M$ and $y = Ax$. Let $R_\alpha$ be a single-valued regularizer. Show that for any $\varepsilon > 0$ there are $\delta > 0$ and $\alpha > 0$ such that if $\|y^\bullet - y\|_Y < \delta$, then $\|R_\alpha y^\bullet - x\|_X < \varepsilon$.

We observe that in this exercise $\delta$ and $\alpha$ generally depend on $x$.

The result of Exercise 2.2.3 is valid also for many-valued regularizers.

So in principle, given a regularizer, we can solve equation (2.0) in a stable way. We are left with two important questions: how to construct regularizers and how to estimate the convergence rate. In the next section we will show a variational method for finding $R_\alpha$ for many correctness sets $X_M$, and then we prove that a stability estimate for the initial equation (2.0) implies some convergence rate for regularization algorithms.

Let us consider Example 2.1.3. In a general situation we cannot expect uniqueness of a solution. For many particular kernels we obtain important equations, and then it is possible to show uniqueness and (which is normally more difficult) to find a stability estimate. If we consider the operators of convolution then in terms of the Fourier transforms uniqueness means that $\hat{k}$ is not zero on any subset of nonzero measure, which is the case when this function is real-analytic on $\mathbb{R}^n$.

For equations with the Riesz type kernels and nonintersecting $\bar{\Omega}$, $\bar{\Omega}_1$, there is uniqueness in $X = L_2(\Omega)$, provided that $\beta \neq 2k$, $\beta \neq 2k + 2 - n$ for any $k = 0, 1, \ldots$, where $n$ is the dimension of the space (see the book [Is4], p. 79).

If $n = 3$ and $\beta = -1$, then there is nonuniqueness even in $C_0^\infty(\Omega)$.

**Exercise 2.2.4.** Show that if $f = \Delta\phi$, where $\phi$ is a $C^2$-function, $\phi = 0$ outside $\Omega$, then $Af(x) = 0$ when $x$ is not in $\Omega$, provided that $n = 3$ and $K(x, y) = |x - y|^{-1}$.

Since in this case the problem has very important applications (inverse gravimetry), it is interesting to find $X_M$ where a solution is unique. This is not a simple (and not completely resolved) question. Referring to Section 4.1 and to the book [Is4], sections 3.1–3.3, we claim that a solution $f$ is unique at least in the following two cases: (1) when $\partial_n f = 0$ on $\Omega$ or (2) $f$ is the characteristic function $\chi(D)$ of a star-shaped (or $x_n$-convex) subdomain of $\Omega$. We will discuss uniqueness in more detail in Section 4.1. Stability is an even more complicated topic. It is quite well understood for the inverse gravimetric problem, and there are some results for the Riesz-type potentials in the paper of Djatlov [Dj].

A convolution equation (2.1.5) can be studied in terms of the function $k_C$, which is defined as $\inf|\hat{k}(\xi)|$ over $|\xi| < C$.

**Exercise 2.2.5.** Assume that $\|f\|_{(1)} \leq M$. Show that a solution $f$ to the convolution equation (2.1.5) satisfies the following estimate: $\|f\|_2 \leq \|F\|_2/k_C + M/C$. By minimizing the right side with respect to $C$, derive from this estimate the logarithmic-type estimate

$$\|f\|_2 \leq M(3T/2\ln B)^{-1/2}(3(2\ln B)^{-1} + 1) \text{ where } B = 1/3(M^2\|F\|_2^{-2}T^{-1})^{1/3}$$

for the Gaussian kernel $k(x) = \exp(-x^2T/2)$.

{*Hint*: Solve the equation for the minimum point, and bound this point from below using the inequality $te^t < e^{2t}$.}

The result of this exercise gives a stability estimate for a solution to the backward heat equation. When $\|F\|_2$ goes to zero, $B$ goes to $+\infty$, and $\|f\|_2$ converges to zero at a logarithmic rate.

## 2.3 Construction of regularizers

We describe a quite general method of a so-called stabilizing functional suggested by Tikhonov [TiA].

We call $\mathcal{M}$ a stabilizing functional for the correctness class $X_M$ if

(2.3.1) $\mathcal{M}$ is a lower-semicontinuous (on $X$) nonnegative functional defined on $X_M$;

(2.3.2) the set $X_{M,\tau} = \{x \in X_M : \mathcal{M}(x) \leq \tau\}$ is bounded in $X$ for any number $\tau$.

We construct a regularizer by using the following minimization problem:

$$\min(\|Av - y\|_Y^2 + \alpha\mathcal{M}(v)) \quad \text{over } v \in X_M. \tag{2.3.3}$$

**Lemma 2.3.1.** *Under the additional condition that $X_{M,\tau}$ is compact in $X$ for any $\tau$, a solution $R_\alpha(y)$ to the minimization problem* (2.3.3) *exists, and $R_\alpha$ is a regularizer.*

PROOF. Let $x_\bullet \in X_M$. We define $\tau = \|Ax_\bullet - y\|_Y^2 + \alpha\mathcal{M}(x_\bullet)$. According to the condition (2.3.2), the set $X_\bullet = \{\mathcal{M}(v) \le \tau/\alpha\} \cap X_M$ is compact, so the lower-semicontinuous functional $\Phi(v; y) = \|Av - y\|_Y^2 + \alpha\mathcal{M}(v)$ has a minimum point $x_*$ on $X_\bullet$. The value $\Phi(x_*)$ is minimal over $X_M$ because if $\Phi(v) \le \Phi(x_\bullet)$, then $v \in X_\bullet$. The set of all minimum points is closed in $X$ due to the semicontinuity of $\Phi$. We denote this set by $x(\alpha)$ or by $R_\alpha(y)$.

The next step is a proof of continuity of $R_\alpha$ for any fixed $\alpha$. Let us assume that it is not continuous at $y$. Then there is a sequence $y_k$ converging to $y$ and $\varepsilon > 0$ such that $d(R_\alpha y_k, R_\alpha y) > \varepsilon$. According to the definition of the distance, we have $x_k \in R_\alpha y_k$ such that $\|x_k - x\|_X > \varepsilon$ for any $x \in R_\alpha y$. Let $x_\bullet \in X_M$. We define $\tau$ as $\sup \Phi(x_\bullet; y_k)$ with respect to $k$. Since the $y_k$ are convergent and $\Phi$ is continuous with respect to $y$, this sup is finite. As above, we have $x_k \in X_\bullet$, which is a compact set, so by extracting a subsequence we can assume that the $x_k$ converge to some $x_\infty \in X_M$. We have $\Phi(x_k; y_k) \le \Phi(v; y_k)$ for any $v \in X_M$. Since $\Phi$ is lower semicontinuous with respect to $x_k$ and continuous with respect to $y_k$, we can pass to the limit and obtain the same inequality with $y$ instead of $y_k$, and $x_\infty$ instead of $x$. This means that $x_\infty$ is a minimum point for $\Phi$ on $X_M$, so it is contained in $R_\alpha y$. On the other hand, $\|x_\infty - x\|_X \ge \varepsilon$ for any $x \in R_\alpha y$, and we arrived at a contradiction.

Now we will show that the $x(\alpha)$ converge to $x$ when $y = Ax$, provided that $\alpha$ goes to 0. Assuming the opposite, we can find a sequence of points $x_k \in x(\alpha_k)$, $\alpha_k < 1/k$ whose distances to $x$ are greater than some $\varepsilon$. Since $y = Ax$ and $\|A(x_k) - y\|_Y^2 + \alpha_k\mathcal{M}(x_k) \le \alpha_k\mathcal{M}(x)$, we conclude that the $x_k$ are contained in the set $X_*$, defined as $\{v : v \in X_M, \mathcal{M}(v) \le \mathcal{M}(x)\}$. Since $X_*$ is compact, by extracting a subsequence we can assume that the $x_k$ converge to $x_*$. By continuity of the distance function we have $x \ne x_*$. On the other hand, by the definition of minimizers we have

$$\|Ax_k - Ax\|_Y^2 \le \alpha_k\mathcal{M}(x) \le (1/k)\mathcal{M}(x),$$

so using continuity of $A$ and passing to the limit we obtain $Ax_* = Ax$. By the uniqueness property we get $x_* = x$, which is a contradiction. Our initial assumption was wrong, and the convergence of $x(\alpha)$ to $x$ is proven. □

We observe that for linear operators $A$, convex sets $X_M$, and strongly convex functionals $\mathcal{M}$ the variational regularizers are single-valued operators, so everything above can be understood in a more traditional sense. Indeed, under these more restrictive assumptions the functional (2.3.3) to be minimized is convex, so a minimum point is unique. The variational construction is not only possible way to find regilarizers, and there is a very important question about an optimal and natural choice of regularization that agrees with intuition and that allows one to improve convergence by using more information about the problem (in the form of constraints).

For linear operators $A$ and Hilbert spaces $X$ and $Y$ we have a somehow stronger result. For references about convex functionals and weak convergence we refer to the book of Ekeland and Temam [ET]. For example, we will make use of the fact that a convex lower-semicontinuous function in $X$ is lower semicontinuous with respect to weak convergence in $X$.

**Lemma 2.3.2.** *If $X$, $Y$ are Hilbert spaces, $x^0 \in X$, and $\mathcal{M}(v) = \|v - x^0\|_X^2$, then the minimization problem* (2.3.3) *has a unique solution $R_\alpha(y)$ that is a regularizer of equation* (2.0).

PROOF. The functional $\Phi(x) = \|Ax - y\|_Y^2 + \alpha\mathcal{M}(x)$ is convex and continuous in $X$.

Let $x_m$ be a minimizing sequence such that $\Phi(x_m) \to \Phi_*$, which is an infimum over $X$. Then $\alpha\|x_m - x^0\|_X \leq C$, so $\{x_m\}$ is bounded. In any Hilbert space bounded closed sets are weakly compact, so we can assume that $\{x_m\}$ is weakly convergent to $x$. It is known [ET] that $\Phi$ is lower semicontinuous with respect to weak convergence, hence we have $\Phi_* = \liminf \Phi(x_m) \geq \Phi(x)$. Since $\Phi_*$ is $\inf \Phi$ over $X$, we have $\Phi(x) = \Phi_*$.

The uniqueness of $x$ as well as continuity will follow form the next example, where we will show that $R_\alpha(x)$ is given by the right side of (2.3.4), which is a linear continuous operator in $X$.

The proof that $x(\alpha)$ (strongly) converges to $x$ when $\alpha \to 0$ is the subject of Exercise 2.3.4.

The proof is complete. □

EXAMPLE 2.3.3. Let $X, Y$ be Hilbert spaces with the scalar products $(\ ,\ )_X, (\ ,\ )_Y$. Assume that $A$ is a linear compact operator from $X$ to $Y$. Let $\mathcal{M}(x) = \|x - x^0\|_X^2$. We let $X = X_M$. In this case a necessary condition for a minimum point $x(\alpha)$ of the quadratic functional $q(v) = (Av - y, Av - y)_Y + \alpha(v - x^0, v - x^0)_X$ is $d/dt(q(x(\alpha) + tu)) = 0$ at $t = 0$ for any $u \in X$. Calculating this derivative, we obtain

$$2(Au, Ax(\alpha) - y)_Y + 2\alpha(u, x(\alpha) - x^0)_X = 2(u, A^*A + \alpha x(\alpha) - \alpha x^0 - A^*y)$$

by the definition of the adjoint operator $A^*$. Since this derivative must be zero for all $u \in X$, we conclude that $x(\alpha)$ is a solution to the equation

$$(A^*A + \alpha)x(\alpha) = A^*y + \alpha x^0. \tag{2.3.4}$$

Lemma 2.3.2 guarantees the convergence of $x(\alpha)$ to a solution $x$ of equation (2.0) when $\alpha \to 0$, provided that $y$ is convergent to $Ax$. It is easy to observe that the operator $A^*A$ is positive and self-adjoint in $X$, so we have uniqueness of a solution $x(\alpha)$ of equation (2.3.4) and therefore uniqueness of the minimizer obtained in Lemma 2.3.3.

**Exercise 2.3.4.** Show that a solution $x(\alpha)$ to the equation (2.3.4) exists and is unique. Show that $R_\alpha y$ defined as $x(\alpha)$ is a regularizer to equation (2.0) on $X = X_M$, provided that the equation $Ax = 0$ has only the zero solution.

{*Hint*: to show that $R_\alpha$ is a regularizer, first consider compact $A$ and make use of eigenfunctions of $A^*A$. The general case can be studied by using more general results about spectral representation of self adjoint operators in a Hilbert space.}

Singular value decomposition is a useful theoretical and computational tool in general, and in solving ill-posed problems in particular. We recall that if $A$ is a compact linear operator from a Hilbert space $X$ into a Hilbert space $Y$, then the operator $A^*A$ is compact self-adjoint, and therefore it has a complete orthonormal system $a_k$ of eigenvectors (functions) corresponding to (nonnegative) eigenvalues $\lambda_k^2$,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq \lambda_{k+1} > 0.$$

We assume here that the equation $Ax = 0$ has only the zero solution. We let $b_k = \|Aa_k\|_Y^{-1} Aa_K$. From these definitions we have

$$A^*Aa_k = \lambda_k^2 a_k,\ AA^*b_k = \lambda_k^2 b_k,\ A^*b_k = \lambda_k a_k,\ Aa_k = \lambda_k b_k. \tag{2.3.5}$$

Observe that the system $\{b_k\}$ is orthonormal. Indeed,

$$(b_k, b_m)_Y = \|Aa_k\|_Y^{-1}\ \|Aa_m\|_Y^{-1}(Aa_k, Aa_m)_Y,$$

and the last factor is equal to $(A^*Aa_k, a_m)_X = \lambda_k^2(a_k, a_m)_X = 0$ when $k \neq m$. It is obvious that the $Y$-norm of any $b_k$ is equal to 1.

**Lemma 2.3.5.** *A necessary and sufficient condition of existence of a solution to equation* (2.0) *is given by the following Picard test:*

$$\sum_{k=1}^{\infty} \lambda_k^{-2} |(y, b_k)_Y|^2 < \infty. \tag{2.3.6}$$

PROOF. Assume that a solution $x$ to (2.0) exists. Then $A^*Ax = A^*y$. Let us calculate the scalar product of both parts of this equality and $a_k$. We obtain

$$(A^*Ax, a_k)_X = (A^*y, a_k)_X = (y, Aa_k)_Y = \lambda_k(y, b_k)_Y$$

from the definitions of $b_k$ and of the adjoint operator. Since $\{a_k\}$ is an orthonormal basis in $X$, we can write $x$ as the sum of the convergent series of $(x, a_k)_X a_k$ and

$$\|x\|_X^2 = \sum_{k=1}^{\infty} |(x, a_k)_X|^2 = \sum_{k=1}^{\infty} \lambda_k^{-4} |(x, A^*Aa_k)_X|^2 = \sum_{k=1}^{\infty} \lambda_k^{-2} |(Ax, b_k)_Y|^2,$$

where we used (2.3.5) and again the definition of the adjoint operator. Since $Ax = y$, we obtain (2.3.6).

On the other hand, let us assume (2.3.6). Let

$$x = \sum_{k=1}^{\infty} \|Aa_k\|_Y^{-1} (y, b_k)_Y a_k.$$

The convergence of the series

$$\|Aa_k\|_Y^{-2}|(y, b_k)_Y|^2$$

follows from condition (2.3.6) and the equality

$$\|Aa_k\|_Y^2 = (Aa_k, Aa_k)_Y = (A^*Aa_k, a_k)_X = \lambda_k^2.$$

Therefore, $x \in X$. By using the definition of $b_k$, it is not difficult to understand that $Ax = y$.

The proof is complete. □

Due to significant increase of computational power computation of the singular-value decomposition for an interesting operator is now quite realistic task, the Picard test is currently becoming a tool for practical solution of inverse problems. Particular numerical methods include range tests which are described in section 10.5. The main difficulty with with this test for strongly ill-posed problems is the very fast (exponential) decay of singular values when $k$ goes to infinity combined with errors in data.

**Exercise 2.3.6.** By using the singular value decomposition, show that for any compact operator $A$ we have $\|x(\alpha)\|_X \le 2^{-1}\alpha^{-1/2}\|y\|_Y$ for any solution $x(\alpha)$ to the regularized equation (2.3.4).

If $A$ itself is positive and self-adjoint, then equation (2.0) can be regularized by the similar equation

$$(A + \alpha)x(\alpha) = y. \tag{2.3.7}$$

**Exercise 2.3.7.** Prove that a solution $x(\alpha)$ to equation (2.3.7) exists for any $y \in Y$, is unique, and $\|x(\alpha)\|_X \le \alpha^{-1}\|y\|_X$.

To be more particular, we consider the integral operator from Example 2.1.3. We let $X = L_2(\Omega)$ and $Y = L_2(\Omega_1)$. Then the adjoint operator

$$A^*F(y) = \int_{\Omega_1} K(x, y)F(x)dx$$

and the regularization of the first kind integral equation is the Fredholm integral equation

$$\alpha f(z) + \int_{\Omega_1} K(x, z)\int_{\Omega} K(x, y)f(y)dy\,dx = \int_{\Omega_1} K(x, z)F(x)dx,\ z \in \Omega,$$

which has a unique solution $f = f(\ , \alpha)$ for any $\alpha > 0$ according to Exercise 2.3.3.

When $\Omega = \Omega_1$ and $K(x, y) = K(y, x)$, the integral operator is self-adjoint in $X$, and one can use the simpler regularization (2.3.7).

EXAMPLE 2.3.8 (SMOOTHING STABILIZERS). Let $\Omega = \Omega_1 = (a, b)$ be an interval of the real axis $\mathbb{R}$, and $X = Y = L_2(a, b)$. Let us consider

$$\mathcal{M}f = \int_\Omega (f^2 + f'^2).$$

From the well-known results about $L_2$-norms and about compactness in Lebesgue spaces (see, e.g., the book of Yosida [Yo], ch. X., sect 1) we derive the properties (2.3.1) and (2.3.2). This functional justifies the term *stabilizing functional*, because for any $\alpha$ we are looking for solutions of our equation with bounded first-order derivatives, which makes the solution stable by smoothing cusps of its graph.

For a smoothing stabilizer, the regularized equation is more complicated. To derive it we can repeat the argument from Example 2.3.3, letting $x(\alpha) = f$ and replacing the new stabilizing term $v$ by $f + tu$. Then, instead of the term $(u, f)_x$, we obtain

$$\int_\Omega 2fu + 2f'u' = 2\int_\Omega (fu - f''u) + \cdots,$$

where ... denotes boundary integrals containing $f'u$ that are the result of integration by parts. Using all $u$ that are zero on $\partial\Omega$, we conclude as above that the factor of $u$ in these integrals must be zero, or

$$-\alpha f''(z) + \alpha f(z) + \int_\Omega K(x, z) \int_\Omega K(x, y) f(y) dy\, dx$$
$$= \int_\Omega K(x, z) F(x)\, dx,\ z \in \Omega,$$

which is an integrodifferential equation for $f$.

In the many-dimensional case one can similarly make use of the regularizer

$$\mathcal{M}f = \int_\Omega (|f|^2 + \nabla f \cdot \nabla f),$$

which is known to improve numerics when a solution to be found is smooth.

As we already noted, the simpler regularizer (2.3.4) works quite well for linear problems, so the smoothing stabilizer is more appropriate for nonlinear problems, which we will not discuss here in detail, referring rather to the books of Engl, Hanke, and Neubauer [EnHN] and of Tikhonov and Arsenin [TiA].

EXAMPLE 2.3.9 (MAXIMAL ENTROPY REGULARIZER). Quite useful are regularizers of physical origin. When one is solving the integral equation from Example 2.1.3 in the class of positive functions $f$, a regularized variational method to solve the equation $Af = F$ is to minimize

$$\|Af - y^\delta\|_Y^2 + \alpha E(f), \quad \text{where } E(f) = \int_\Omega f \ln(f/m),$$

where $m$ is some positive function reflecting a priori information about $f$. When $n = 1$ and $\Omega = (0, l)$, one often chooses $m(x) = 1/x$. The functional $-E(f)$ is known as Shannon entropy, which is a measure of the informational content of $f$.

While this functional looks different from the $L_2$-norm, it can be transformed into this norm by a (nonlinear) Nemytsky operator $f \to f_*$ defined by the pointwise relation $f \log(f/m)(x) = f_*^2(x)$, so that the above regularization theory can be applied.

Another important example of regularazing functional is given by the bounded variation described in more detail in section 10.1.

## *Quasi-solutions*

Let $X_\bullet$ be a compact set in $X$. An element $x_\bullet$ of this set is called a quasi-solution to the equation (2.0) with respect to $X_\bullet$ if $x_\bullet$ is a solution to the minimization problem

$$\min \|Av - y\|_Y \text{ over } v \in X_\bullet. \tag{2.3.8}$$

Since $X_\bullet$ is a compact set, there is a solution to this problem.

**Exercise 2.3.10.** Let $\mathfrak{X}(y)$ be the set of all solutions to (2.3.8). Prove that $\mathfrak{X}(y)$ is continuous with respect to $y$ in the sense of the distance defined in Section 2.2.

Now we consider a regularization method based on approximation of $X_M$ by compact subsets and first suggested by V. Ivanov (see the book of Ivanov, Vasin, and Tanana [IvVT]).

Let $X(k)$ be compact subsets of $X_M$. Assume that $X(k+1) \neq X(k) \subset X(k+1)$ and that closure of the union of $X(k)$ over $k$ is $X_M$. A typical example of $X(k)$ is the intersection of $X_M$ and of the ball of radius $k$ of the finite-dimensional space span $\{x(1), \ldots, x(k)\}$ generated by vectors $x(1), \ldots, x(k)$ in $X$.

We will show that the sets $\mathfrak{X}(y_\delta, k)$ that consist of all quasi-solutions with respect to $X(k)$ are convergent to $\{x\}$ when $\|y_\delta - y\|_Y < \delta, \delta \to 0$ and $k \to \infty$. Indeed, let $x_k \in \mathfrak{X}(y_\delta, k)$. Since the $X(k)$ approximate $X_M$ in the above sense, there are $x_k^\bullet \in X(k)$ such that $\|x_k^\bullet - x\|_X = \omega_X(k, x)$ converge to zero as $k \to \infty$. By the minimizing property,

$$\|Ax_k - y_\delta\|_Y \le \|Ax_k^\bullet - y_\delta\|_Y \le \|Ax_k^\bullet - Ax\|_Y + \|Ax - y_\delta\|_Y \le \omega_A(k, x) + \delta,$$

where $\omega_A(k, x) = \|Ax_k^\bullet - Ax\|_Y$ goes to zero as $k \to \infty$ by continuity of $A$. By the triangle inequality,

$$\|Ax_k - Ax\|_Y \le \|Ax_k - y_\delta\|_Y + \|y_\delta - y\|_Y < 2\delta + \omega_A(k, x).$$

By property (2.2.2) of $X_M$, we have

$$\|x_k - x\|_X \le \omega(2\delta + \omega_A(k, x)), \tag{2.3.9}$$

where $\omega$ is a stability estimate for (2.0) at the point $x$. Since $x_k$ is an arbitrary point of $\mathfrak{X}(y_\delta, k)$, we have an estimate of the distance between this set and the exact solution $x$.

Summing up, we have a regularization algorithm $R_\alpha y = \mathfrak{X}(y, k)$, where $\alpha$ is $1/k$. Here $\mathfrak{X}(k, y)$ is the set of all solutions to the minimization problem (2.3.8) with $X_\bullet = X(k)$.

This regularization can be used practically for a numerical solution of equation (2.0). Normally, one uses as $\{x(k)\}$ trigonometric or polynomial bases in standard spaces. At each step of the algorithm one has to solve a complicated (as a rule nonconvex) optimization problem (2.3.8).

**Exercise 2.3.11.** Formulate the minimization problem for the solution of equation (2.0) where $A$ is the operator from Example 2.1.1 (backward heat equation at the initial moment of time) by using the given basis. From the estimate in this example derive an estimate (2.3.0) for this particular problem with a particular function $\omega$.

## 2.4 Convergence of regularization algorithms

Let $\omega$ be a stability estimate for equation (2.0) on the compact set $X_\bullet$, which is defined as $X_M \cap \{\mathcal{M}(x) \leq \tau/\alpha\}$, where $\tau = \|Ax_\bullet - y\|_Y^2 + \alpha\mathcal{M}(x_\bullet)$ for certain $x_\bullet \in X_M$, $\mathcal{M}(x) \leq \mathcal{M}(x_\bullet)$. Such a stability estimate exists as soon as we have uniqueness of a solution in $X_M$. We observe that any stability estimate on $X_M$ will be a stability estimate on its subset $X_\bullet$, but for the set $X_\bullet$ we can expect a better stability.

**Lemma 2.4.1.** *If $R_\alpha$ is a variational regularization algorithm from Section* 2.3, *then*

$$\|x^\bullet(\alpha) - x\|_X \leq \omega(2\|Ax - y\|_Y + (\alpha\mathcal{M}(x))^{1/2}) \tag{2.4.1}$$

*for any $x^\bullet(\alpha)$ in the minimizing set $x(\alpha)$.*

PROOF. We have

$$\|Ax^\bullet(\alpha) - y\|_Y^2 \leq \|Ax^\bullet(\alpha) - y\|_Y^2 + \alpha\mathcal{M}(x^\bullet(\alpha)) \leq \|Ax - y\|_Y^2 + \alpha\mathcal{M}(x)$$

according to the definition of $x(\alpha)$ by minimization. Therefore,

$$\|Ax^\bullet(\alpha) - y\|_Y \leq \|Ax - y\|_Y + (\alpha\mathcal{M}(x))^{1/2}.$$

By the triangle inequality,

$$\|Ax^\bullet(\alpha) - Ax\|_Y \leq \|Ax^\bullet(\alpha) - y\|_Y + \|Ax - y\|_Y \leq 2\|Ax - y\|_Y + (\alpha M(x))^{1/2},$$

and inequality (2.4.1) follows from the definition of the stability estimate $\omega$. □

The estimate (2.4.1) suggests the following choice of the regularization parameter. Suppose our data $y_\delta$ are given with the error $\delta : \|Ax - y_\delta\|_Y \leq \delta$ and we have the a priori information

$$\mathcal{M}(x) \leq M. \tag{2.4.2}$$

Then we can choose $\alpha(\delta) = \delta^2/M$ and obtain the estimate

$$\|x^\bullet(\alpha) - x\|_X \leq \omega(3\delta). \tag{2.4.3}$$

Since the stability of minimization procedure (2.3.4) deteriorates when $\alpha \to 0$, we cannot tell whether this estimate reflects the actual complexity of the problem or whether the choice of the regularization parameter is optimal. Examples of stability estimates from Section 2.2 suggest that in many cases one can expect logarithmic convergence rates.

What is missing in this discussion is how to solve the minimization problem (2.3.3). A constructive way to do it in the linear case is to solve equation (2.3.4). We discuss this problem in more detail assuming that $A$ is a compact linear operator in a Hilbert space $X$.

First, let $A$ be a self-adjoint, positive, and compact operator. We write the equation (2.3.7) as

$$(\alpha + A)(x + x^\alpha + x^\delta) = y + (y^\delta - y),\ y = Ax,\ \|y_\delta - y\|_Y < \delta, \tag{2.4.4}$$

where $x^\delta$ solves the equation $(\alpha + A)x^\delta = y_\delta - y$. It is easy to see that

$$\|x^\delta\|_X \leq \delta/\alpha \tag{2.4.5}$$

and that $(\alpha + A)(x + x^\alpha) = y$, or $(\alpha + A)x^\alpha = -\alpha x$. The self-adjoint, compact operator $A$ has an orthonormal basis of eigenvectors $a_k$ corresponding to eigenvalues $\lambda_k$. We can assume $\lambda_k \geq \lambda_{k+1}$. Moreover, let $X$ be the uniqueness set, so $\lambda_k > 0$. We set $x_k = (x, a_k)_X$. Then the equation for $x^\alpha$ is $(\alpha + \lambda_k)x_k^\alpha = -\alpha x_k$, and we have

$$\|x^\alpha\|_X^2 \leq \sum_{k \leq K} \alpha^2(\alpha + \lambda_k)^{-2} x_k^2 + \sum_{K<k} x_k^2 \leq \alpha^2 \lambda_K^{-2} \|x\|_X^2 + \sum_{K<k} x_k^2. \tag{2.4.6}$$

This estimate and the estimate (2.4.5) in principle guarantee convergence of our regularization procedure at this particular $x$. Indeed, choose $K$ large enough to make the sum over $k > K$ small; then choose $\alpha$ small to make the first term small, and then choose small $\delta$.

To obtain estimates, we consider the set $X_M$ defined by the inequalities

$$|x_k| \leq m_k \tag{2.4.7}$$

and let $M_K$ be $\left(\sum_{K<k} m_k^2\right)^{1/2}$. We assume that the series is convergent. Then $M_K$ goes to zero as $K$ goes to infinity. From (2.4.6) we have $\|x^\alpha\|_X^2 \leq \alpha^2\lambda_K^{-2}M_1^2 + M_K^2$, so $\|x^\alpha\|_X \leq \alpha\lambda_K^{-1}M_1 + M_K$. Now, from (2.4.5) we have the following estimate:

$$\|x(\alpha) - x\|_X \leq \alpha M_1/\lambda_K + M_K + \delta/\alpha, \tag{2.4.8}$$

and then the optimal choice of $\alpha$ depends on our constraint (2.4.7).

We consider Example 2.1.1 again. If we let $X = L_2(0,1)$, replace $x$ by $u_0$, and assume $\|u_0\|_X^2 + \|\partial_x u_0\|_X^2 \leq C_1$, then by using the Fourier expansion with respect to the trigonometric basis, we conclude that $|x_k| \leq Ck^{-1}$, so we let $m_k = Ck^{-1}$.

**Exercise 2.4.2.** Show that for the solution $u(\alpha)$ to the regularized equation $(\alpha + A)u(\alpha) = u_{T\delta}$ to the backward heat equation $Au_0 = u_T$, $u_0 \in X_M$, we have the error estimate

$$\|h(\alpha) - h\|_X \le \alpha^{1/2} C_1 e^{\pi^2 K^2 T} + CK^{-1/2} + \delta/\alpha.$$

Show that by an appropriate choice of $K$ and $\alpha$ one can achieve the logarithmic estimate

$$C_2/(-\ln \delta)$$

of the error with respect to $\delta$.

{*Hint*: as in Exercise 2.2.5, minimize with the respect to $K$ and then with respect to $\alpha$.}

A similar scheme can be developed for the regularized equation (2.3.4). Cheng and Yamamoto [CheY1] proposed to replace a minimizer $x(\alpha)$ of the regularized discrepancy functional (2.3.2) by any element $x(\alpha, \varepsilon)$ which a $\varepsilon$-approximate minimizer, i.e.

$$\|Ax(\alpha, \varepsilon) - y\|_Y^2 + \alpha\mathcal{M}(x(\alpha, \varepsilon) \le inf(\|Ax - y\|_Y^2 + \alpha\mathcal{M}(x)) + \varepsilon^2$$

where *inf* is over $x \in X_M$. They showed that a stability estimate on $X_M$ implies a convergence rate similar to rate of Lemma 2.4.1. Although, theoretically compactness assumption is removed, in practical numerical solutions it does not remove the question about how to find an approximate infimum.

An important issue is the choice of the regularization parameter $\alpha$. There is the well-known discrepancy principle suggested by Morozov: let

$$\|Ax(\alpha(\delta)) - y^\delta\|_Y = \delta. \tag{2.4.9}$$

This is an a posteriori method because it only says that the parameter $\alpha$ we have chosen was consistent with the accuracy $\delta$ of the data $y$.

**Lemma 2.4.3.** *Equation* (2.4.9) *has a unique solution* $\alpha(\delta)$, *and* $x(\alpha(\delta))$ *converges to* $x$ *in* $X$ *when* $\delta \to 0$.

PROOF. We will show that the left side of (2.4.9) is increasing with respect to $\alpha$. Observe that using the expansion $y^\delta = \sum y_k^\delta b_k$ and the relations (2.3.5) we obtain

$$\begin{aligned} Ax(\alpha) &= \sum A(\alpha + A^*A)^{-1} A^*(y_k^\delta b_k) = \sum A(\alpha + A^*A)^{-1} y_k^\delta \lambda_k a_k \\ &= \sum \lambda_k(\alpha + \lambda_k^2)^{-1} y_k^\delta A a_k = \sum \lambda_k^2(\alpha + \lambda_k^2)^{-1} y_k^\delta b_k. \end{aligned}$$

It suffices to prove monotonicity for the squared norm, which is

$$(Ax(\alpha) - y^\delta, Ax(\alpha) - y^\delta)_Y = \sum_{k=1}^{\infty} \alpha^2(\alpha + \lambda_k^2)^{-2} y_k^{\delta 2}.$$

It is clear that any term of the last series is increasing with respect to $\alpha$ unless it is zero.

To prove convergence we can use the variational formulation (2.3.3) of equation (2.3.4). Then

$$\|Ax(\alpha) - y^\delta\|_Y^2 + \alpha\|x(\alpha)\|_X^2 \le \|Ax - y^\delta\|_Y^2 + \alpha\|x\|_X^2 = \delta^2 + \alpha\|x\|_X^2.$$

Since due to our choice of $\alpha$ in (2.4.9) the first term is $\delta^2$, we conclude that $\|x(\alpha)\|_X^2 \le \|x\|_X^2$. Hence

$$\begin{aligned}\|x(\alpha) - x\|_X^2 &= \|x(\alpha)\|_X^2 - 2(x(\alpha), x)_X + \|x\|_X^2 \\ &\le 2\|x\|_X^2 - 2(x(\alpha), x)_X,\end{aligned}$$

and convergence $x(\alpha) \to x$ in $X$ follows from convergence of the second term to $2\|x\|_X^2$. We will show this convergence.

According to (2.3.4) and the equality $Ax = y$, we have

$$x(\alpha) - x = (\alpha + A^*A)^{-1}A^*(y^\delta - y) - \alpha(\alpha + A^*A)^{-1}x,$$

or in coordinates as above,

$$(x(\alpha) - x)_k = \lambda_k(\alpha + \lambda_k^2)^{-1}(y^\delta - y)_k - \alpha(\alpha + \lambda_k^2)^{-1}x_k.$$

For any fixed $k$ the eigenvalue $\lambda_k > 0$. Besides, $\alpha(\delta) \to 0$ when $\delta \to 0$, so $(x(\alpha) - x)_k \to 0$. It suffices to show convergence to 0 of

$$(2.4.10) \qquad (x(\alpha) - x, x)_X = \sum^{K}(x(\alpha) - x)_k x_k + \sum_{K+1}(x(\alpha) - x)_k x_k.$$

Let $\varepsilon > 0$. Fix $K$ such that

$$\sum_{K+1} x_k^2 \le \varepsilon^2(16\|x\|_X^2)^{-1}.$$

By the Cauchy-Schwarz inequality the second term on the right side of (2.4.10) is not greater than

$$\|x(\alpha) - x\|_X \left(\sum_{K+1} x_k^2\right)^{1/2} \le (\|x(\alpha)\|_X + \|x\|_X)\varepsilon(4\|x\|_X)^{-1} \le \varepsilon/2$$

due to the inequality $\|x(\alpha)\|_X \le \|x\|_X$. Since $K$ is fixed, the first term on the right side of (2.4.10) goes to 0 when $\delta$ goes to 0 because of convergence of coordinates.

The proof is complete. □

One cannot obtain convergence rates without using stability estimates for the original equation (2.0).

Maslov [Mas] first observed that for some regularization algorithms, convergence is equivalent to the existence of a solution to equation (2.0). Consider the regularizer (2.3.4). Assume that $R(A)$ is dense in $Y$ and $\ker A = \{0\}$ (i.e., we have uniqueness of a solution). These assumptions are quite realistic and are satisfied in many applications.

Let $\delta \to 0$ (so the $y_\delta$ converge to $y$) and $\alpha(\delta) \to 0$ and assume that the $x(\alpha)$ converge to $x_\bullet$. Since all operators in (2.3.4) are continuous, we can pass to the limit and obtain $A^*Ax_\bullet = A^*y$. Since $R(A)$ is dense in $Y$, we have $\ker A^* = \{0\}$, so $Ax_\bullet = y$, and we have a solution to equation (2.0). On the other hand, if a solution to (2.0) exists, then Exercises 2.3.3 and 2.2.3 guarantee convergence of this regularization algorithm.

In the paper of Maslov [Mas] the same result is obtained for the "adjoint" regularization $x^*(\alpha) = A^*(\alpha + AA^*)^{-1}y(\delta)$.

**Exercise 2.4.4.** Show that for this regularization, convergence of $x^*(\alpha)$ when $\alpha$ and $\delta$ go to 0 implies that the original equation (2.0) has a solution $x \in X$, provided that the solution is unique and the range of $A$ is dense in $Y$.

## 2.5 Iterative algorithms

To solve the regularized equations of the previous section in practice, one uses various iterative methods. At any step of such an algorithm one solves a well-posed problem, and if this algorithm is convergent, one obtains an approximate solution of the original problem, avoiding long and expensive minimization procedures like (2.3.3), which are even more complicated by the nonconvexity of functional, which is typical for inverse problems in partial differential equations.

We start with a short description of these algorithms with iterative solutions of the linear regularized equation (2.3.4). The most minimization algorithms make use of the gradient methods, which require that one minimizes the functional $\Phi(v) = \|Av - y\|_X^2 + \alpha\mathcal{M}(v)$ by iterations:

$$x(m+1) = x(m) - 1/2\tau(m)\Phi'(x(m)), \tag{2.5.1}$$

where $\tau(m)$ is a parameter that may not depend on $m$, and $\Phi'(x)$ is a gradient of the functional $\Phi$ at $x$. We will start with the simplest case of a linear operator $A$ in a Hilbert space $X$ and the functional $\mathcal{M}(v) = \|v - x^0\|_X^2$. Then $\Phi'(x) = 2(A^*Ax - A^*y + \alpha(x - x^0))$, and the relations (2.5.1) are

$$x(m+1) = x(m) - \tau(m)((A^*A + \alpha I)x(m) - A^*y - \alpha x^0). \tag{2.5.2}$$

To analyze the expected convergence we will assume that $\tau(m) = \tau$ and make use of the singular-value decomposition (2.3.5), denoting by $x_k$ the $k$th coordinate of $x$ in the basis $\{a_k\}$ and similarly for $y_k$ in the basis $\{b_k\}$.

**Lemma 2.5.1.** *Let $\|y^\delta - Ax\|_Y \le \delta$. Let $x(1), x^0 \in X$. Then under the condition*

$$0 < \tau(\|A\|^2 + \alpha) < 1 \tag{2.5.3}$$

*we have the following estimate:*

$$\begin{aligned} \|x(m+1) - x\|_X^2 \le 4(1 - \tau\lambda_K^2)^{2m}(\|x(1)\|_X^2 + \|x\|_X^2 + \|x^0\|_X^2) \\ + \delta^2\alpha^{-1} + 4\alpha^2\lambda_K^{-4}\|x - x^0\|_X^2 + R(K), \end{aligned} \tag{2.5.4}$$

*where $R(K) = 4\sum_{K<k}(x_k^2(1) + x_k^2 + x_k^{02} + (x - x^0)_k^2)$.*

PROOF. From (2.5.2) we have

$$x_k(m+1) = q_k x_k(m) + \tau(A^* y^\delta + \alpha x^0)_k, \quad q_k = 1 - \tau(\lambda_k^2 + \alpha).$$

By induction it is easy to check that

$$\begin{aligned} x_k(m+1) &= q_k^m x_k(1) + (q_k^{m-1} + \cdots + 1)\tau(A^* y^\delta + \alpha x^0)_k \\ &= q_k^m x_k(1) + (1 - q_k^m)/(1 - q_k)\tau(\lambda_k^2 x_k + \lambda_k w_k^\delta + \alpha x_k^0), \end{aligned}$$

because $A^* y^\delta = A^* A x + A^* w^\delta$, $w^\delta = y^\delta - y$ and because of the relations (2.3.5). Using the definition of $q_k$ we obtain

$$\begin{aligned} x_k(m+1) - x_k &= q_k^m x_k(1) + (1 - q_k^m)\lambda_k(\lambda_k^2 + \alpha)^{-1} w_k^\delta \\ &\quad - q_k^m(\lambda_k^2 + \alpha)^{-1}(\lambda_k^2 x_k + \alpha x_k^0) + \alpha(\lambda_k^2 + \alpha)^{-1}(x^0 - x)_k, \end{aligned} \tag{2.5.5}$$

where $w_k^\delta$ are coefficients with respect to $b_k$ and by using the inequalities

$$\begin{aligned} (a + b + c + d)^2 &\le 4(a^2 + b^2 + c^2 + d^2), \\ (\lambda_k^2 + \alpha)^{-2}(\lambda_k^2 x_k + \alpha x_k^0)^2 &\le \max\{x_k^2, x_k^{02}\} \le x_k^2 + x_k^{02} \end{aligned}$$

we conclude that

$$\begin{aligned} (x_k(m+1) - x_k)^2 &\le 4 q_k^{2m} x_k^2(1) + \alpha^{-1} w_k^{\delta 2} \\ &\quad + 4 q_k^{2m}(x_k^2 + x_k^{02}) + 4\alpha^2(\lambda_k^2 + \alpha)^{-2}(x - x^0)_k^2, \end{aligned}$$

where we have utilized that $\lambda_k(\lambda_k^2 + \alpha)^{-1} \le 2^{-1}\alpha^{-1/2}$. Summing over $k$ and using (2.5.3) we obtain

$$\begin{aligned} &\|x(m+1) - x\|_X^2 \\ &\le 4\left(\sum_{k \le K} q_k^{2m}(x_k^2(1) + x_k^2 + x_k^{02}) + \sum_{K<k}(x_k^2(1) + x_k^2 + x_k^{02})\right) \\ &\quad + \alpha^{-1}\delta^2 + 4\sum_{k \le K}\alpha^2\lambda_k^{-4}(x - x^0)_k^2 + 4\sum_{K<k}(x - x^0)_k^2, \end{aligned}$$

which gives the bound (2.5.4). □

Lemma 2.5.1 implies convergence of the iterations (2.5.2), provided that $\delta \to 0$. Indeed, since the series for the coordinates of $x, x(1)$, and $x^0$ are convergent, for any $\varepsilon > 0$ we can find $K$ such that $R(K) < \varepsilon/4$. Fix this $K$ and then fix $\alpha > 0$ such that the third term on the right side of (2.5.4) is less than $\varepsilon/4$. After that, find $\delta$ such that the second term is less than $\varepsilon/4$, and $m$ such that the first term is less than $\varepsilon/4$. In many applied problems $e^{-CK} < \lambda_K < e^{-cK}$, while a smoothness assumption on $x, x(1), x^0$ implies that $R(K) < CK^{-1}$. Then we have to choose $K \sim \varepsilon^{-1}$ and then $\lambda_K \sim e^{-C\varepsilon^{-1}}$, so we must let $\alpha \sim e^{-C\varepsilon^{-1}}$ and $\delta \sim e^{-C\varepsilon^{-1}}$. Since the iterations (2.5.2) constitute a gradient method for minimization of a (coercive for fixed $\alpha$) convex functional, it is natural that for any fixed $\alpha$ they converge.

**Exercise 2.5.2.** Show that for fixed $\alpha > 0$ the iterations $x(m)$ determined by (2.5.2) converge in $X$ to $(\alpha + A^*A)^{-1}(A^*y + \alpha x^0)$.

Sometimes (2.5.2) with $\alpha = 0$ and $x^0 = 0$ is called the Landweber method:

$$x(m+1) = (I - \tau A^*A)x(m) + \tau A^*y.$$

The biggest advantage of iterative algorithms shows up in solving nonlinear equations, because in this case variational regularization as a rule leads to a non-convex optimization problem with many possible minima and local minima.

A natural analogy of the Landweber method for the nonlinear equation (2.0) is the iterative method

$$x(m+1) = x(m) + \tau A'^*(x(m))(y - A(x(m))),$$

where $A'(x)$ is the Fréchet derivative of the operator $A$ at a point $x$. There are convergence proofs, provided that the initial guess $x(1)$ is sufficiently close to a solution $x$.

Another type of iterative algorithm suitable for ill-posed problems is the conjugate gradients method

$$\begin{aligned} x(m+1) &= x(m) + (w(m), v(m))_X (v(m), v^*(m))_X^{-1} v(m), \\ w(m) &= A^*y - A^*Ax(m), \; v^*(m) = A^*Av(m), \\ v(m+1) &= -w(m) - (w(m), v^*(m))_X (v(m), v^*(m))_X^{-1} v(m), \end{aligned}$$

with some choice of $x(1)$ and $v(1) = y - A^*Ax(1)$.

There are proofs of convergence of the conjugate gradient method as well.

These gradient-type methods have the advantage that no inversion of $A'$, $A'*$ is needed. This is especially important when solving ill-posed problems, because this inversion is not stable and is computationally expensive. However, gradient methods are relatively slow, and sometimes one makes use of Newton type methods like

$$\begin{aligned} x(m+1) = x(m) &- (\alpha(m) + A'^*A')^{-1}) \\ &\times (A'^*(Ax(m) - y) + \alpha(m)(x(m) - x^0)). \end{aligned} \tag{2.5.6}$$

We refer for convergence results and other iterative algorithms to the book of Engl, Hanke, and Neubauer [EnHN]. Besides a need for a good initial guess, the fundamental difficulty with convergence of the Newton type algorithms (2.5.6) is that the Frechet derivative $A'$ is not invertible in standard functional spaces when the equation (2.0) represents severely ill-posed problem. We call the problem represented by the equation (2.0) mild ill-posed if $A'$ is invertible from one Sobolev space into another. In case of mild ill-posedness there is a deep and interesting method of Nash-Moser which guarantees local convergence by combining iterations with some smoothing operators [CN], [Ham]. This method is not applicable to severely ill-posed problems. The best result about convergence of the iterations (2.5.6) is due to Hohage [Ho] and we briefly decribe his findings.

To guarantee convergence it was assumed in [EnHN], [Ho] that the initial guess $x(0)$ satisfies the source type condition

$$x(0) - x = f(A'^*(x)A'(x))v, \tag{2.5.7}$$

where $v \in X$ has "small" norm and $f$ is some continuous function on the spectrum of $A'^*(x)A'(x)$. For mildly ill-posed problems $f(\lambda) = \lambda^\kappa$, $0 < \kappa < 1$ and for typical severely ill-posed inverse problem in partial differential equations one expects only logarithmic stability and then it is more natural to choose $f(\lambda) = (-ln\lambda)^{-p}$. Since the operator $A$ (and hence $A'$) is highly smoothing the source condition (2.5.7) is much stronger that standard "smallness" condition for convergence of Newton type methods. Replacing power by logarithm is certainly less restrictive. Another condition in [Ho] for convergence of (2.5.6) is the assumption that the range of the $A'(x)$ is not changing dramatically with $x$. More precisely, one assumes that for $x_1, x_2$ which are close to $x$ there are continuous linear operators $R(x_1, x_2)$, $Q(x_1, x_2)$ from $Y$ into itself and from $X$ into $Y$ and constants $C(R)$, $C(Q)$ such that

$$A'(x_1) = R(x_1, x_2)A'(x_2) + Q(x_1, x_2) \tag{2.5.8}$$

with

$$\|R(x_1, x_2)\| \leq C(R), \;\; \|Q(x_1, x_2)\| \leq C(Q)\|A'(x)(x_1 - x_2)\|_Y.$$

So far the both conditions could not be checked for important severely ill-posed problems, like the inverse gravimetry problem or inverse scattering problem for obstacles which are domains. The condition (2.5.8) seems to be especially delicate since singularities of solutions dictating range of $A'(x)$ depend on shape of unknown domain. It looks that there is a better chance to satisfy conditions (2.5.7), (2.5.8) in problems of determining coefficients of boundary conditions for elliptic problems when distance to singularities is given.

# 3

# Uniqueness and Stability in the Cauchy Problem

In this chapter we formulate and in many cases prove results on uniqueness and stability of solutions of the Cauchy problem for general partial differential equations. One of the basic tools is Carleman-type estimates. In Section 3.1 we describe the results for a simplest problem of this kind (the backward parabolic equation), where a choice of the weight function in Carleman estimates is obvious, and the method is equivalent to that of logarithmic convexity. In Section 3.2 we formulate general conditional Carleman estimates and their simplifications to second order equations, and we apply the results to the general Cauchy problem and give numerous counterexamples showing that the assumptions of positive results are quite sharp. We also formulate a global version of Holmgren's theorem and the recent result of Tataru on nonanalytic equations. In Section 3.3 we consider elliptic and parabolic equations of second order, construct for them pseudo-convex weight functions and obtain complete and general uniqueness and stability results for the Cauchy problem. Section 3.4 is devoted to substantially less understood hyperbolic equations and Schrödinger-type equations. Here, for some particular but interesting domains we also give appropriate weight functions and obtain a quite explicit description of uniqueness domains for lateral Cauchy problems. We analyse the increased stability in the Cauchy problem which could has important consequences for numerical algorithms. Additional information to Sections 3.2–3.4 can be found in the book of Zuily [Z]. In section 3.5 we expose recent results on Carleman estimates, uniqueness and stability in the Cauchy problem for systems of partial differential equations, obtaining relatively complete results for isotropic Maxwell's and linear elasticity systems. We emphasize that these quesitons for more general anisotropic systems are barely touched in research, except probably some "generic" cases of Cauchy problems with simple characteristics.

By $C$ we denote generic constants depending only on $A, \mathbf{A}, \Omega, \Gamma, \varphi$ which are introduced later at appropriate sections. Any additional dependence will be specified in parentheses.

## 3.1 The backward parabolic equation

We are interested in finding a solution $u$ to the evolution equation

$$\partial_t u + Au = 0 \quad \text{on } \Omega \times (0, T), \tag{3.1.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T) \tag{3.1.2}$$

given the final data

$$u = u_T \quad \text{on } \Omega \times \{T\}. \tag{3.1.3}$$

### *Method of eigenfunctions*

To illustrate the basic ideas and methods, we first consider the simple case of $A = -\operatorname{div}(a\nabla) + c$, where $a, c$ are measurable bounded functions on $\Omega$, $0 < \epsilon \leq a$, $0 \leq c$. The results, however, are not trivial, even for the one-dimensional heat equation $\partial_t u - \partial_x^2 u = 0$ when $\Omega = (0, 1)$.

The operator $A$ with the Dirichlet boundary conditions (3.1.2) is self-adjoint in the Hilbert space $L^2(\Omega)$ with the domain $H_{(2)} \cap \mathring{H}_{(1)}(\Omega)$ when we assume in addition that $a \in C^1(\overline{\Omega})$, so there is a complete orthonormal system of eigenfunctions $e_k(x)$ of $A$ with eigenvalues $\lambda_{k+1} \geq \lambda_k \geq 0$, $k = 1, 2 \ldots$. For any solution of (3.1.1), (3.1.2) we have

$$u(t, x) = \sum u_k(t) e_k(x). \tag{3.1.4}$$

This function solves (3.1.1), (3.1.2) if and only if the coefficients $u_k$ satisfy the following ordinary differential equations:

$$\partial_t u_k + \lambda_k u_k = 0 \quad \text{when } 0 < t < T, u_k(0) = u_{0k},$$

where $u_0 = u(0)$. Solving them, we obtain $u_k(t) = u_{0k} e^{-\lambda_k t}$, so

$$u(t, x) = \sum u_{0k} e^{-\lambda_k t} e_k(x). \tag{3.1.5}$$

The backward uniqueness easily follows from this representation, because the coefficients $u_{0k}$ are uniquely determined from the relations $u_{0k} e^{-\lambda_k T} = u_{Tk}$, which follow from (3.1.3) and (3.1.5). Since the $\lambda_k$ behave like $Ck^p$ with some $p > 0$ for large $k$ (e.g., for the one-dimensional heat equation and $\Omega = (0, 1)$ we have $e_k(x) = \sqrt{2}\sin(k\pi x)$ and $\lambda_k = \pi^2 k^2$). These relations show that for any $u_T \in L_2(\Omega)$ (and even $H_{(m)}(\Omega)$) a solution $u$ does not exist, and when it does, it is exponentially unstable: for $u$ that is the $k$th term of the sum (3.1.5) with $u_{0k} = \varepsilon e^{\lambda_k T}$ we have $\|u_0\|_2(\Omega) = \varepsilon e^{\lambda_k T}$, while $\|u_T\|_2(\Omega) = \varepsilon$.

The next natural question is how to compute $u$ (or $u_0$), provided that it exists. The first answer was obtained in the pioneering paper of Fritz John [Jo1]. We describe a couple of more recent algorithms.

## *Quasi-reversibility*

The method of quasi-reversibility has been suggested in the book of Lattes and Lions [LL].

We replace (3.1.1) by the regularized "higher"-order equation

$$\partial_t u(;\alpha) + Au(;\alpha) - \alpha A^2 u(;\alpha) = 0. \tag{3.1.6}$$

Again, by using eigenfunctions we obtain for the coefficients $u_k(;\alpha)$ the ordinary differential equations

$$\partial_t u_k(;\alpha k) + (\lambda_k - \alpha\lambda_k^2)u_k(;\alpha k) = 0, 0 < t < T, u_k(T;\alpha) = u_{Tk}.$$

Solving them we get

$$u(t;\alpha) = \sum u_{Tk} e^{(\lambda_k - \alpha\lambda_k^2)(T-t)} e_k(x). \tag{3.1.7}$$

Since the $\lambda_k$ go to infinity as $k$ increases, this series is convergent in $L_2(\Omega)$ for any $u_T$ in this space and for any $t < T$. Moreover, when $\alpha$ goes to 0, the regularized solutions $u_\alpha$ are convergent to the solution $u$ (in $L_2(\Omega)$ for any $t \in (0, T)$), provided that $u$ exists.

## *Regularization by pseudo-parabolic equations*

Another example of a (weakly) convergent numerical algorithm is due to Gajewski and Zacharias [GZ]. They suggested regularizing equation (3.1.1) by the equation

$$\partial_t(u_\alpha - \alpha A u_\alpha) + Au_\alpha = 0.$$

**Exercise 3.1.1.** Prove the existence of solutions of the pseudo-parabolic regularization and their convergence to $u$ when $u$ exists.

## *Method of logarithmic convexity*

The next natural question is about stability in the backward evolutionary equations and about the rate of convergence of numerical algorithms. It is not a simple question unless one makes use of a priori bounds and logarithmic convexity.

To show that the logarithm $F(t)$ of the squared norm of a solution $f(t) = \|u(t)\|_2^2(\Omega)$ is a convex function, we use again the simple but important example of $A = -\operatorname{div}(a\nabla) + c$ assuming that $a, c, -\partial_t a, -\partial_t c$ are measurable, bounded, and nonnegative, and that $a \geq \varepsilon_0 > 0$.

Now we will prove that

$$F'' \geq 0. \tag{3.1.8}$$

We have

$$F' = f'/f, \qquad F'' = (ff'' - (f')^2)/f^2.$$

By using the definition of the $L_2$-norm and the differential equation (3.1.1) we obtain

$$f' = 2\int_\Omega u\partial_t u = 2\int_\Omega u(\operatorname{div}(a\nabla u) - cu) = 2\int_\Omega (-a|\nabla u|^2 - cu^2),$$

where we have used integration by parts and the boundary condition (3.1.2). Further,

$$\begin{aligned} f'' &= 2\int_\Omega (-2a\nabla u \cdot \partial_t \nabla u - \partial_t a|\nabla u|^2 - 2cu\partial_t u - \partial_t cu^2) \\ &\geq \int_\Omega (4\operatorname{div}(a\nabla u)\partial_t u - 4cu\partial_t u) = 4\int_\Omega (\partial_t u)^2 = 4\|\partial_t u\|_2^2, \end{aligned}$$

where we integrated by parts again, used the conditions on $a, c$, and expressed $\partial_t u$ from the differential equation (3.1.1). Sometimes we will drop, for brevity, the symbol $\Omega$ in norms and scalar products. Now, we have

$$(f''f - (f')^2) \geq 4\|\partial_t u\|_2^2\|u\|_2^2 - (2(u, \partial_t u)_2)^2 \geq 0$$

according to the Schwarz inequality.

We have proved that $F$ is convex. Therefore, $F(t) \leq (1 - t/T)F(0) + t/TF(T)$, or

$$f(t) \leq f(0)^{1-t/T} f(T)^{t/T} \quad \text{when} \quad 0 \leq t \leq T.$$

Using the definition of $f$ and the final data, we get

$$\|u(t)\|_2(\Omega) \leq M^{1-t/T}\|u_T\|_2^{t/T}(\Omega) \tag{3.1.9}$$

under the constraint $\|u(t)\|_2(\Omega) \leq M$. It is easy to understand that this estimate is sharp. It can be considered a conditional stability estimate (under the a priori bound $\|u(t)\|_2 \leq M$).

**Exercise 3.1.2.** Prove that under the additional constraint $\|\partial_t u\|_2(\Omega) \leq M_1$ on $(0, T)$ one has

$$\|u(0)\|_2^2(\Omega) \leq -MM_1T(1 - \ln(-M_1T/(M\ln\epsilon))/\ln\epsilon,$$
$$\text{where } \epsilon = \|u_T\|_2(\Omega)/M.$$

{*Hint*: use Taylor's formula for $\|u(t)\|_2^2$ around $t = 0$; then bound $\|u(0)\|_2^2$ by using (3.1.9) and the constraint on $\partial_t u$. Minimize the bound with respect to $t$.}

Now we will prove a quite general result on logarithmic convexity and therefore stability in the backward evolution equations. This result is a weaker version of the theorem of Agmon and Nirenberg [AN].

Let $A = A(t)$ be a linear operator in the (complex) Hilbert space $H$ with domain $D(t)$. By $\|\ \|$, $(\ ,\ )$ we denote respectively the norm and the scalar product in $H$. We consider the following generalization of equation (3.1.1):

$$\|\partial_t u + Au\| \leq \alpha\|u\| \qquad \text{on } (0, T). \tag{3.1.10}$$

We assume that

$$A = A_+ + A_-, \tag{3.1.11}$$

where $A_+$ is a linear symmetric operator in $H$ with domain $D(t)$ and $A_-$ is a skew-symmetric operator. Moreover, they satisfy the following conditions:

$$\|A_- u\|^2 \le \alpha(\|A_+ u\|\|u\| + \|u\|^2) \tag{3.1.12}$$

and

$$\partial_t (A_+ u, u) \le 2\Re(A_+ u, \partial_t u) + \alpha(\|A_+ u\|\|u\| + \|u\|^2) \tag{3.1.13}$$

for some constant $\alpha$.

**Theorem 3.1.3.** *Let $u(t) \in D(t)$, $u \in C^1([0, T]; H)$ be a solution of the differential inequality* (3.1.10)*, where $A$ satisfies the conditions* (3.1.11) – (3.1.13)*.*
*Then*

$$\|u(t)\| \le C_1 \|u(0)\|^{1-\lambda} \|u(T)\|^{\lambda}$$

*with* $C_1 \le \exp((2\alpha + 2)T + 2e^{CT}/C)$ *and* $\lambda = (1 - e^{-Ct})/(1 - e^{-CT})$ *or* $(e^{C(t-T)} - e^{-CT})/(1 - e^{-CT})$ *with $C$ depending on $\alpha$. In addition, when $\alpha = 0$ we can take $C_1 = 1$ and $\lambda = t/T$.*

In the proof we will use two elementary lemmas. We set $q = \|u\|^2$, $\psi = 2\Re(f, u)/q$, and $l = \ln q - \int_0^t \psi$.

**Lemma 3.1.4.** *There is a constant $C \ge 0$ depending on $\alpha$ such that $\partial_t^2 l + C|\partial_t l| + C > 0$ on $(0, T)$. In addition, $C$ can be taken as $0$ when $\alpha = 0$.*

PROOF. Let $f = \partial_t u + Au$.
We have

$$\partial_t q = 2\Re(u, \partial_t u) = -2\Re(A_+ u, u) + 2\Re(f, u) = -2\Re(A_+ u, u) + \psi q,$$

where $\psi = 2\Re(f, u)/q$, and we have used that due to skew-symmetry, $\Re(A_- u, u) = 0$. Observe also that for a symmetric operator $A_+$ we have $\Im(A_+ u, u) = 0$. So

$$\partial_t l = \partial_t q/q - \psi = -2(A_+ u, u)/q$$

and

$$\begin{aligned}
\partial_t^2 l &= -2\partial_t (A_+ u, u)/q + 2(A_+ u, u)(-2(A_+ u, u) + \psi q)/q^2 \\
&\ge -4\Re(A_+ u, \partial_t u)/q - 2\alpha\|A_+ u\|\|u\|/q - 2\alpha \\
&\quad -4(A_+ u, u)^2/q^2 + 2(A_+ u, u)\psi/q \\
&= 4(\|A_+ u\|^2 - (A_+ u, u)^2/q)/q + 4\Re(A_+ u, A_- u)/q \\
&\quad -4\Re(A_+ u, f)/q - 2\alpha\|A_+ u\|/\|u\| - 2\alpha \\
&\quad +4(A_+ u, u)\Re(f, u)/q^2.
\end{aligned}$$

To obtain the inequality we have used condition (3.1.13) and then condition (3.1.11).

We set $(A_+u, u) = \|A_+u\|\|u\|\theta$. We represent $A_+u$ as $\beta u + u^\perp$, where $u^\perp$ is orthogonal to $u$. By scalar multiplication, we find $\beta = (A_+u, u)/\|u\|^2$, and then by the Pythagorean theorem we get $\|u^\perp\|^2 = \|A_+u\|^2(1-\theta^2)$. Since $\Re(A_-u, u) = 0$ due to skew-symmetry, we conclude that

$$\begin{aligned}-4\Re(A_+u, A_-u) &= -4\Re(u^\perp, A_-u) \le 4\|u^\perp\|\|A_-u\| \\ &\le 2(\|u^\perp\|^2 + \|A_-u\|^2) \\ &\le 2(\|A_+u\|^2(1-\theta^2) + \alpha\|A_+u\|\|u\| + \alpha\|u\|^2)\end{aligned}$$

due to the Schwarz inequality, our formula for $u^\perp$, and condition (3.1.12). Summing up and using the Schwarz inequality and the inequality $\|f\| \le \alpha\|u\|$ several times, we obtain

$$\begin{aligned}\partial_t^2 l &\ge 4\|A_+u\|^2 q^{-1}(1-\theta^2) - 2\|A_+u\|^2 q^{-1}(1-\theta^2) - 2\alpha\|A_+u\|/\|u\| \\ &\quad - 2\alpha - 4\alpha\|A_+u\|/\|u\| - 2\alpha\|A_+u\|/\|u\| - 2\alpha - 4\alpha\|A_+u\|/\|u\| \\ &= 2\sigma^2(1-\theta^2) - 12\alpha\sigma - 4\alpha\end{aligned}$$

when we set $\sigma = \|A_+u\|/\|u\|$.

Observe that $|\partial_t l| = 2|\theta|\sigma$. If $\theta^2 < \frac{1}{4}$, then

$$\partial_t^2 l \ge \sigma^2 - 12\alpha\sigma + 36\alpha^2 - 36\alpha^2 - 4\alpha \ge -36\alpha^2 - 4\alpha.$$

If $\frac{1}{4} \le \theta^2 \le 1$, then $|\partial_t l| \ge |\sigma|$ and

$$\partial_t^2 l \ge -12\alpha\sigma - 4\alpha, \text{ so } \partial_t^2 l + 12\alpha|\partial_t l| + 4\alpha \ge 0.$$

In the both cases we have the required inequality with $C = \max\{36\alpha^2 + 4\alpha, 12\alpha\}$.

The proof is complete. □

**Lemma 3.1.5.** *Under the conditions of Theorem* 3.1.3 *we have*

$$\begin{aligned}\ln\|u(t)\| &\le \ln\|u(0)\|(e^{-\gamma t} - e^{-\gamma T})/(1 - e^{\gamma T}) \\ &\quad + \ln\|u(T)\|(1 - e^{-\gamma t})/(1 - e^{-\gamma T}) + C_2,\end{aligned}$$

*where $\gamma$ is either $C$ or $-C$, $C_2 \le (2\alpha + 2)T + 2e^{CT}/C$, and $C_2 = 0$ when $\alpha = 0$.*

PROOF. Let $L$ be a solution to the differential equation $\partial_t^2 L + C|\partial_t L| + C = 0$ on $(0, T)$ coinciding with $l$ at the endpoints $0, T$. The existence of $L$ can be proven by using a priori estimates on $L, \partial_t L$ that follow from the observation that $L$ satisfies the linear differential equation with the coefficient $C \operatorname{sign} \partial_t L$ of $\partial_t L$. By subtracting the equation for $L$ from the inequality for $l$ given by Lemma 3.1.4, we conclude that $\partial_t^2(l - L) + C|\partial_t(l - L)| \ge 0$. Then $l - L \le 0$ by maximum principles, and it suffices to bound $L$. From extremum principles it follows that the function $L(t)$ cannot achieve a (local) minimum on $(0, T)$, so $\partial_t L > 0$ on $(0, \tau)$, and $\partial_t L < 0$ on $(\tau, T)$ for some $\tau$ between 0 and $T$. Therefore, $L$ satisfies the linear differential equation $\partial_t^2 L + C_\tau \partial_t L + C = 0$, where $C_\tau$ is $C$ on $(0, \tau)$ and $-C$ outside the interval.

We will first bound the solution $v$ to the differential equation $\partial_t^2 v + C_\tau \partial_t v = 0$ on $(0, T)$ coinciding with $l$ at the endpoints. Then $w = L - v$ solves the inhomogeneous equation and has zero boundary data.

Since $\partial_t v$ solves a linear first-order homogeneous ODE, it does not change its sign. Let $l(0) \leq l(T)$. Then $\partial_t v \geq 0$. Consider the solution $V$ to the ODE $\partial_t^2 V + C\partial_t V = 0$ with the same boundary data. We have

$$\partial_t^2(v - V) + C\partial_t(v - V) = (C - C_\tau)\partial_t v \geq 0,$$

and $v - V$ is zero at the endpoints. By the maximum principles, $v \leq V$. The function $V$ is easy to calculate by solving the linear ordinary differential equation with constant coefficients and satisfying the boundary conditions. It is

$$l(0)(e^{-Ct} - e^{-CT})/(1 - e^{-CT}) + l(T)(1 - e^{-Ct})/(1 - e^{-CT}).$$

When $l(0) > l(T)$ we obtain a similar bound with $C$ replaced by $-C$.

To bound $w$ we will solve the two ODE for $w : \partial_t^2 w + C\partial_t w + C = 0$ on $(0, \tau)$ and $\partial_t^2 w - C\partial_t w + C = 0$ on $(\tau, T)$ and we will satisfy zero boundary data at $0, T$ and the continuity conditions for $w, \partial_t w$ at $t = \tau$ from the left and from the right. After standard calculations we obtain

$$\begin{aligned} w(t) &= (2\tau - T + 2C^{-1} - 2C^{-1}e^{C\tau})(e^{Ct} - e^{CT})/(e^{CT} - 2e^{C\tau} + e^{2C\tau}) \\ &\quad - T + t \text{ when } \tau < t, \\ w(t) &= (T - 2\tau + 2C^{-1} - 2C^{-1}e^{C(T-\tau)})(1 - e^{-Ct})/(-e^{C(T-2\tau)} + 2e^{-C\tau} - 1) \\ &\quad - t \text{ when } t < \tau. \end{aligned}$$

The numerator and denominator of $w$ in the first case increase in $\tau$, replacing the first one by its maximal absolute value $\frac{2}{C}(e^{CT} - 1) - T$ and the second one by its minimal absolute value $e^{CT} - 1$, we bound $w$ on $(\tau, T)$ by $2e^{CT}/C - T + t$. Similarly, on $(0, \tau)$ we have the bound $2e^{CT}/C - t$. Replacing these two functions by their maximal values (at $t = 0$ and $t = T$) we obtain the same bound $2e^{CT}/C$).

In sum, we have

$$\begin{aligned} l \leq L = v + w &\leq l(0)(e^{-\gamma t} - e^{-\gamma T})/(1 - e^{-\gamma T}) \\ &\quad + l(T)(1 - e^{-\gamma t})/(1 - e^{-\gamma T}) + 2e^{CT}/C, \end{aligned}$$

where $\gamma$ is $C$ or $-C$. From the definitions of $l$ and $f$ we have $\ln q \leq l + 2T\alpha$ and $l \leq \ln q + 2T\alpha$. In addition, the factors of $l(0)$ and of $l(T)$ in the above inequality are between 0 and 1. In sum, we conclude that $\ln q(t) \leq l(t) + 2T\alpha \leq \cdots + 4T\alpha$, where $\ldots$ denotes the terms with $l(0)$, $l(T)$ replaced by $\ln q(0)$, $\ln q(T)$, and the first claim follows.

When $\alpha = 0$, the function $l$ is $\ln q$ and it is convex by Lemma 3.1.4. This completes the proof. □

PROOF OF THEOREM 3.1.3. The needed estimate follows from the inequality of Lemma 3.1.5 by taking exponents of both parts. When $\alpha = 0$ this lemma also implies that $C_1 = \exp C_2$ is 1. Letting $\alpha$ go to 0, we conclude from Lemma 3.1.4 that $C$ goes to 0, and calculating the limit of $\lambda$, we conclude the proof. □

This theorem can be applied to a second-order parabolic equation with the Dirichlet or Neumann lateral boundary data and even to "parabolic" equation changing direction of time.

EXAMPLE 3.1.6 (A PARABOLIC EQUATION OF SECOND ORDER). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^2$. Let

$$Au = -\sum \partial_k(a_{jk}\partial_j u) + \sum b_j\partial_j u + cu \tag{3.1.14}$$

with the domain $D(t) = \mathring{H}_{(1)}(\Omega) \cap H_{(2)}(\Omega)$. We assume that $a_{jk}$, $\partial_t a_{jk}$, $b$, $\operatorname{div} b \in C^1(\overline{\Omega} \times [0, T])$, $c$, $\partial_t c \in L_\infty(\Omega \times [0, T])$. Here $b = (b_1, \ldots, b_n)$.

We will check conditions (3.1.11) – (3.1.13) with

$$A_+u = -\sum \partial_k(a_{jk}\partial_j u) + \left(-\frac{1}{2}\operatorname{div} b + c\right)u,$$

$$A_-u = \sum \partial_j(b_j u) - \frac{1}{2}(\operatorname{div} b)u.$$

It is obvious that $A = A_+ + A_-$. Integrating by parts and using the boundary condition $u = 0$ on $\partial\Omega$, we conclude that $A_+$ is symmetric and $A_-$ is skew-symmetric.

Let us consider condition (3.1.12). The left side is the sum of integrals of $B\partial_j u\partial_k u$, $B\partial_j uu$, $Bu^2$ over $\Omega$, where $B$ is the product of either $b_j$ and $b_k$ or of $\operatorname{div} b$ and $b_j$. Letting $\| \; \| = \| \; \|_2(\Omega)$, we obtain the bound in the first case

$$\int_\Omega B\partial_j u\partial_k u = -\int_\Omega \partial_k(B\partial_j u)u \le C(\|\partial_k\partial_j u\|\|u\| + \|\partial_j u\|\|u\|)$$

by using the Schwarz inequality. Here $C$ depends on $|B|_1$. Since the operator $A_+$ is a second-order elliptic operator and $u$ satisfies zero Dirichlet conditions on $\partial\Omega$, the Schauder-type a priori estimates (Theorem 4.1) imply that $\|\partial_k\partial_j u\| + \|\partial_k u\| \le C(\|A_+u\| + \|u\|)$, and we obtain (3.1.12).

To prove (3.1.13) it is sufficient by using the symmetry of $a_{jk}$ and integration by parts to observe that

$$\partial_t \int_\Omega A_+uu = \sum \int_\Omega \partial_t(a_{jk}\partial_j u\partial_k u) = \int_\Omega \left(\sum \partial_t a_{jk}\partial_j u\partial_k u + 2A_+u\partial_t u\right)$$

and to bound

$$\int_\Omega \partial_t a_{jk}\partial_j\partial_k uu,$$

which can be done as before.

**Exercise 3.1.7 (The Third Boundary Value Condition).** Under the assumptions of Example 3.1.6 with respect to the coefficients of $A$, prove uniqueness of a solution $u$ of the backward initial problem with the lateral boundary conditions

$$\sum a_{jk}\partial_j u\nu_k + bu = 0 \text{ on } \partial\Omega \times (0, T)$$

under the regularity assumptions that $\partial_t u, \partial_j\partial_k u, \partial_t^2 u, \partial_j\partial_k\partial_t u \in L_2(\Omega)$ at any $t \in (0, T)$ and that the corresponding norms are bounded uniformly with respect to $t$.

From the proof of Theorem 3.1.3 and from Example 3.1.6 one can see that to satisfy conditions (3.1.11) – (3.1.13), the operator $A$ must be subordinated to its symmetric part $A_+$. At present it is not clear whether backward uniqueness holds without this assumption when $A$ depends on $t$.

EXAMPLE 3.1.8 (FORWARD-BACKWARD PARABOLIC EQUATION). Let $\Omega$ be the unit interval $(0, 1)$ in $\mathbb{R}$, and $A = -\partial_x(a\partial_x u)$. Let $a, \partial_t a \in C^1(\overline{\Omega} \times [0, T])$. We assume that

$$\text{(3.1.15)} \qquad \text{for any } t \text{ either } \partial_t a \le 0 \text{ on } \Omega \text{ or } a \ge \epsilon \text{ on } \Omega$$

for some positive $\varepsilon$. Then the conditions of Theorem 3.1.3 are satisfied, and therefore we have uniqueness and conditional stability of a solution $u$ to the differential equation $\partial_t u = \partial_x(a\partial_x u)$ on $Q = \Omega \times (0, T)$ given zero lateral boundary data $u = 0$ on $\partial\Omega \times (0, T)$ and the final data. We consider $u$ with $\partial_x^2 u \in L_2(Q)$.

We check conditions (3.1.11) – (3.1.13) with the obvious choice $A_+ = A$. Then condition (3.1.12) is satisfied with any $\alpha$. We will check condition (3.1.13). Let us consider

$$\begin{aligned}
\partial_t(A_+u, u) = \partial_t \int_\Omega (-\partial_x(a\partial_x u))u &= \int_\Omega \partial_t(a\partial_x u\partial_x u) \\
&= \int_\Omega (\partial_t a(\partial_x u)^2 + 2a\partial_x u\partial_t\partial_x u) \\
&\le \int_\Omega ((\partial_t a(\partial_x u)^2 - 2(\partial_x(a\partial_x u))\partial_t u),
\end{aligned}$$

where we integrated by parts with respect to $x$ using the zero lateral boundary conditions. The term with $\partial_t a$ is nonpositive together with $\partial_t a$. If $\partial_t a$ is positive at a point of $\Omega$ for some $t$, we integrate by parts in the first integral again and we obtain that this integral is

$$-\int_\Omega \partial_x(\partial_t a\partial_x u)u \le C\|u\|_{(2)}(\Omega)\|u\| \le \alpha\|Au\|\|u\|$$

due to the Cauchy-Schwarz inequality and condition (3.1.15), because then $a \ge \varepsilon$, the operator $A$ is uniformly elliptic, and $\|u\|_{(2)}(\Omega) \le \alpha\|Au\|$ is the standard elliptic estimate. The second term is $2(Au, \partial_t u)$.

So we have condition (3.1.13)

It is quite interesting that this equation can change type from a forward parabolic equation to a backward parabolic equation at any point of $\Omega$. For simplicity, we considered the one-dimensional case, but nothing will change for equation (3.1.14) with $b = 0$ if in condition (3.1.15) we request nonpositivity of the matrix $(\partial_t a_{jk})$.

## *Use of semigroups*

The idea of using semigroups was systematically developed by Krein and Prozorovskaya [Kre], pp. 73, 81, and we present some results of their theory.

Let us consider the differential equation

$$\partial_t u + \zeta A u = 0 \quad \text{on } (0, T), \tag{3.1.16}$$

where $A$ is a linear closed operator in a Banach Space $X$ with domain $D(A)$ that is dense in $X$, and $\| \ \|_X$ denotes the norm in $X$. A solution $u(t;\zeta)$ of this equation satisfying the initial data $u(0) = u_0$ is a function continuous from $[0, T]$ into $X$ that is differentiable on $(0, T)$, $u(t;\zeta) \in D(A)$, and such that (3.1.16) holds. The basic assumption is that there exists a sector $S = \{|\arg\zeta| < \phi\}$ of the complex plane such that for any $\zeta \in S$ and for any initial data $u_0 \in X$ there is a solution to this initial problem that is complex-analytic with respect to $\zeta \in S$ and that satisfies the following estimate:

$$\|u(1;\zeta)\|_X \le C_\phi e^{\sigma t}\|u_0\|_X, \; t = \Re\zeta. \tag{3.1.17}$$

This estimate follows from the following bound on the operator norm of the resolvent of the operator $A$ : $\|(A - \zeta I)^{-1}\| \le C/|\zeta - \sigma|$ when $\Re\zeta > \sigma$ for some $C$ and $\sigma$. The last condition is possible to check when $A$ corresponds to several elliptic boundary value problems, including the first boundary value problem for higher-order elliptic equations and the elliptic oblique derivative problem for second-order equations. This is not easy analytical work. The most recent reference is to the paper of Colombo and Vespri [CoV], where they considered operators $A$ corresponding to general elliptic boundary value problems for higher-order operators satisfying the complementing and normality conditions under the assumptions that the coefficients of the differential operators are merely continuous and of those the boundary conditions are accordingly $C^k(\overline{\Omega})$-smooth with natural choice of $k$.

Under condition (3.1.17) and under the a priori constraint $\|u(t;1)\|_X \le M$, the following stability estimate is valid:

$$\|u(t)\|_X \le CM^{1-\mu(t)}e^{\sigma t}\|u_T\|_X^{\mu(t)}, \tag{3.1.18}$$

where $\mu(\zeta)$ is the harmonic measure of the cut $[T, \infty)$ in $S$ with respect to the point $\zeta$. This function is defined as a bounded harmonic function of $\zeta$ in $S_T = S\backslash[T, +\infty)$ with the boundary values 0 on $\partial S$ and 1 on the cut. By using conformal mappings onto standard domains and Giraud's extremum principle as in Section 9.3, it is possible to show that $\mu$ exists, is unique, and satisfies the following estimates: $t^{2\phi/\pi}/C < \mu(t)$ when $0 < t < \tau$, and $1 - (T - t)^{1/2}/C < \mu(t)$ when $T - \tau < t < T$ for some $\tau$.

To obtain (3.1.18) we observe that the function $s(\zeta) = \ln\|e^{-\sigma\zeta}u(;\zeta)\|_X$ is subharmonic in $S_T$ as the norm of an analytic function with values in $X$. From (3.1.17) it follows that $s(t) \le \ln C_\phi\|u_T\|_X$ when $T < t < +\infty$. So we have $\liminf s(\zeta) \le (1 - \mu(\zeta))\ln C_\phi M + \mu(\zeta)\ln\|u_T\|_X$ when $\zeta \in \partial S_T$. By the maximum principle this inequality holds when $\zeta \in S_T$, in particular when $\zeta = t \in (0, T)$. Taking exponents of both parts and multiplying by $e^{\sigma t}$ we arrive at (3.1.18).

One-dimensional case was considered in some detail by Watanabe [W].

## 3.2 General Carleman estimates and the Cauchy problem

Let $m$ be a multi-index with positive integer components $m_j$ satisfying the following condition: $m_1 = \cdots = m_q > m_{q+1} \geq \ldots$. We define $\nabla_q$ as $(\partial_1, \ldots, \partial_q, 0, \ldots, 0)$.

We consider the differential operator

$$A(x; \partial) = \sum a_\alpha \partial^\alpha \qquad \text{(the sum is over } |\alpha : m| \leq 1\text{)},$$

where $|\alpha : m| = \alpha_1/m_1 + \cdots + \alpha_n/m_n$. We define its $m$-principal part $A_m$ as the sum of the same terms with $|\alpha : m| = 1$. We introduce the $m$-principal symbol of this operator

$$A(x; \zeta) = \sum a_\alpha i^{|\alpha|} \zeta^\alpha, \qquad |\alpha : m| = 1.$$

We will assume that the $a_\alpha$ are in $L_\infty(\Omega)$ and the coefficients of the $m$-principal part are in $C^1(\overline{\Omega})$. In sections 3.2–3.5, $\Omega$ is a bounded domain in $\mathbb{R}^n$.

Let a function $\varphi$ be in $C^2(\overline{\Omega})$ and $\nabla_q \varphi \neq 0$ on $\overline{\Omega}$. We introduce the exponential weight function $w(x) = \exp(\tau\varphi(x))$.

We remind that by $C$ we denote generic constants depending only on $A, \mathbf{A}, \Omega, \Gamma, \varphi$. Any additional dependence will be specified in parentheses.

**Theorem 3.2.1.** *Suppose that either (a) $A_m(x; \xi) \neq 0$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$ or (b) the coefficients of $A_m$ are real-valued.*

*If the conditions*

$$A_m(x; \zeta) = 0, \qquad \zeta = \xi + i\tau\nabla_q\varphi, \qquad \tau \neq 0 \tag{3.2.1}$$

*imply that*

$$\sum_{j,k \leq q} (\partial_j \partial_k \varphi (\partial A_m/\partial \zeta_j)(\overline{\partial A_m/\partial \zeta_k}) + \tau^{-1} \Im \partial_k A_m (\overline{\partial A_m/\partial \zeta_k})) > \delta \tag{3.2.2}$$

*in $\Omega$ for some positive number $\delta$, then there is a constant $C$ such that*

$$\tau \int_\Omega |\partial^\alpha u|^2 w^2 dx \leq C \int_\Omega |Au|^2 w^2 dx, \ |\alpha : m| < 1 \tag{3.2.3}$$

*when $C < \tau$ for all functions $u$ in $C_0^\infty(\Omega)$.*

The conditions of this theorem on the function $\varphi$ are called strong pseudo-convexity conditions. When $A$ is the Laplace operator ($A(\zeta) = \zeta_1^2 + \cdots + \zeta_n^2$), then these conditions are certainly satisfied for strictly convex functions $\varphi$, but not only for them. When $A$ is the wave operator ($A(\zeta) = \zeta_1^2 + \cdots + \zeta_n^2 - \zeta_{n+1}^2$), this convexity concept is adjusted to space-time geometry, which is discussed in more detail in Section 3.4.

A proof of Theorem 3.2.1 is quite long and technical. General ideas are to use differential quadratic forms and to utilize a energy integrals methods of Friedrichs, Leray, and Lewy from the theory of hyperbolic partial differential equations of higher order. In the isotropic case ($m_1 = \dots m_n$), a proof is given by Hörmander in his book [Hö1], section 8.4, and in the general case by Isakov in the paper [Is12]. Further development of Carleman estimates in the isotropic case can be found in the paper of Nirenberg [Ni]. More recently, Tataru [Tat3] included boundary terms in the isotropic case. There are several applications of Carleman estimates, for example to solvability of the equation $Au = f$ for any (regular) $f$, to uniqueness of propagation of singularities, and to exact and approximate boundary controllability for partial differential equations. In this section we will derive from them a theorem about uniqueness and stability in the general Cauchy problem. These estimates were introduced in 1938 by Carleman [Ca] exactly for this purpose (in a particular case of first order systems in the plane with simple characteristics).

In case of second order operators the conditions of Theorem 3.2.1 can be relaxed.

**Theorem 3.2.1′.** *Let us assume that A is a partial differential operator of second order with real-valued principal coefficients ($m = (2, \dots, 2)$).*

*Let a function $\psi \in C^2(\overline{\Omega})$ and the conditions*

$$A_m(x;\xi) = 0, \ \sum(\partial A_m/\partial\xi_j)\partial_j\psi = 0, \qquad \xi \neq 0, \tag{3.2.1′}$$

*imply that*

$$\sum(\partial_j\partial_k\psi)(\partial A_m/\partial\xi_j)(\partial A_m/\partial\xi_k)$$

$$+\sum((\partial_k\partial A_m/\partial\xi_j)\partial A_m/\partial\xi_k - \partial_k A_m\partial^2 A_m/\partial\xi_j\partial\xi_k)\partial_j\psi > 0 \tag{3.2.2′}$$

*in $\overline{\Omega}$. Moreover, let us assume that*

$$A(x;\nabla\psi(x)) \neq 0, x \in \overline{\Omega},$$

*and let us introduce*

$$\varphi = e^{\sigma\psi}. \tag{3.2.4}$$

*Then there are constants $C_1(\sigma)$, $C_2$ such that*

$$\tau^{3-2|\alpha|}\int_\Omega |\partial^\alpha u|^2 w^2 \leq C\left(\int_\Omega |Au|^2 w^2 + \int_{\partial\Omega}(\tau|\nabla u|^2 + \tau^3|u|^2)w^2\right) \tag{3.2.3′}$$

*when $C_2 < \sigma$, $C_1 < \tau$, $|\alpha| \leq 1$, for all functions $u \in H_{(2)}(\Omega)$.*

The conditions of Theorem 3.2.1′ on function $\psi$ are called the pseudo-convexity conditions.

We will show that for second order operators $A = \sum a_{jk}\partial_j\partial_k$ pseudo-convexity of $\psi$ implies strong pseudo-convexity of $\varphi$ given by (3.2.4) for large $\sigma$. Then Theorem 3.2.1′ follows from results of Tataru [Tat3].

Indeed, from (3.2.4) we have

$$\partial_j\varphi = \sigma\varphi\partial_j\psi,\ \partial_j\partial_k\varphi = \sigma\varphi(\partial_j\partial_k\psi + \sigma\partial_j\psi\partial_k\psi).$$

From these formulae by standard calculations the left side in (3.2.2), $m = (2, \dots, 2)$, $q = n$, is

$$4\sum\sigma\varphi(\partial_j\partial_k\psi + \sigma\partial_j\psi\partial_k\psi)a_{jl}\zeta_l a_{kp}\overline{\zeta}_p + 2/\tau\Im\sum\partial_k a_{jl}\zeta_j\zeta_l a_{kp}\overline{\zeta}_p$$

$$= 4\sigma\varphi\sum(\partial_j\partial_k\psi + \sigma\partial_j\psi\partial_k\psi)a_{jk}a_{kp}(\xi_l\xi_p + \tau^2\sigma^2\varphi^2\partial_l\psi\partial_p\psi)$$

$$+ 2/\tau\Im\sum\partial_k a_{jl}a_{kp}(\xi_j + i\tau\sigma\varphi\partial_j\psi)(\xi_l + i\tau\sigma\varphi\partial_l\psi)(\xi_p - i\tau\sigma\varphi\partial_p\psi)$$

$$= 2\sigma\varphi\Big(2\sum(\partial_j\partial_k\psi + \sigma\partial_j\psi\partial_k\psi)a_{jl}a_{kp}(\xi_l\xi_p + \tau^2\sigma^2\varphi^2\partial_l\psi\partial_p\psi)$$

$$+ \sum\partial_k a_{jl}a_{kp}(-\partial_p\psi\xi_j\xi_k + 2\partial_l\psi\xi_j\xi_p + \tau^2\sigma^2\varphi^2\partial_j\psi\partial_l\psi\partial_p\psi)\Big).$$

We will denote the last expression by $\mathfrak{H}$.

To achieve positivity of $\mathfrak{H}$ we can use homogeneity with respect $(\xi, \tau\sigma\varphi)$ and assume that $|\xi|^2 + \tau^2\sigma^2\varphi^2 = 1$. First we consider $\tau = 0$. Passing to the limit in (3.2.1) as $\tau$ goes to zero we obtain the equalities (3.2.1′). Moreover, standard calculations show that $\mathfrak{H}$ becomes the sum of the left side in (3.2.2′) and of

$$2\sigma^2\varphi\sum\partial_j\psi\partial_k\psi a_{jl}a_{kp}\xi_l\xi_p = 2\sigma^2\varphi\left(\sum a_{jl}\partial_j\psi\xi_l\right)^2.$$

Hence $\mathfrak{H}$ is positive for any $x \in \overline{\Omega}$ and $\xi \neq 0$. By compactness and continuity arguments $C^{-1}\sigma\varphi < \mathfrak{H}$ when $\sigma\tau\varphi < C^{-1}$. Now we consider $C^{-1} < \sigma\tau\varphi$. The sum of the terms of $\mathfrak{H}$ containing the highest power of $\sigma$ is

$$\sigma\sum\partial_j\psi\partial_k\psi a_{jl}a_{kp}(\xi_l\xi_p + \sigma^2\tau^2\varphi^2\partial_l\psi\partial_p\psi)$$

$$= \sigma\left(\left(\sum_{jl}\partial_j\psi\xi_l\right)^2 + \sigma^2\tau^2\varphi^2\left(\sum a_{jl}\partial_j\psi\partial_l\psi\right)^2\right) \geq C^{-1}\sigma^3\tau^2\varphi^2,$$

because by the conditions of Theorem 3.2.1′ $A(x, \nabla\psi(x)) \neq 0$ and because of compactness and continuity reasons. The modulus of the remaining terms in $\mathfrak{H}$ is bounded by $C\sigma\varphi(|\xi|^2 + \sigma^2\tau^2\varphi^2)$. Summing up the above inequalities and using that $|\xi|^2 + \sigma^2\tau^2\varphi^2 = 1$ we conclude that

$$\mathfrak{H} > \sigma\varphi(C^{-1}\sigma^3\tau^2\varphi^2 - C(|\xi|^2 + \sigma^2\tau^2\varphi^2)) > \sigma\varphi(C^{-3}\sigma - C) > 0,$$

provided $\sigma > C^4$.

We mention that for elliptic, parabolic, and hyperbolic operators of second order there are Carleman estimates in Sobolev spaces of negative order $H_{(-1)}$ which are useful when handling equations in the divergent form with reduced regularity of solutions [Im], [IIY], [IY2]. In [IIY] these estimates are derived from the Carleman

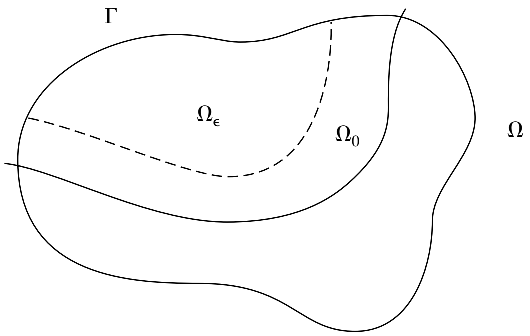


FIGURE 3.1.

estimate (3.2.1) replacing $u$ by a cut-off function multiplied by a negative power of the Laplacian with the parameter $\tau$ and using appropriate commutators bounds for pseudo-differential operators.

As a first application of Carleman estimates we obtain uniqueness and stability results for the following Cauchy problem:

$$Au = f \text{ on } \Omega, \qquad \partial_\nu^j u = g_j,\ j \le m_1 - 1 \text{ on } \Gamma,$$
$$\partial^\alpha u \in L^2(\Omega) \text{ when } |\alpha : m| < 1. \tag{3.2.5}$$

Here $\Gamma$ is a part of $\partial\Omega$ of the class $C^{m_1}$ that is open in $\partial\Omega$. We define $\Omega_\epsilon$ as $\Omega \cap \{\varphi > \varepsilon\}$. We illustrate our problem in Figure 3.1

**Theorem 3.2.2 (Uniqueness and Stability).** *Let $\varphi$ be a function satisfying the conditions of Theorem* 3.2.1. *Let us assume that $\varphi < 0$ on $\partial\Omega \setminus \Gamma$.*

*Then there are constants $C, \kappa \in (0, 1)$ depending on $\Omega, \Gamma, \varphi$ and $\varepsilon$ such that for a solution $u$ to the Cauchy problem* (3.2.5) *we have*

$$\|\partial^\alpha u\|_2(\Omega_\varepsilon) \le C(F + M^{1-\kappa} F^\kappa) \text{ when } |\alpha : m| < 1, \tag{3.2.6}$$

*where $F$ is $\|f\|_2(\Omega) + \sum \|g_j\|_{(m_1-j-1/2)}(\Gamma)$ (the sum is over $j \le m_1 - 1$), and $M$ is the sum of $\|\partial^\alpha u\|_2(\Omega)$ over $\alpha$ with $|\alpha : m| < 1$.*

This theorem guarantees uniqueness of $u$ on $\Omega_0$, provided that we are able to find a strongly pseudo-convex function $\varphi$ that agrees with $\Omega$ and $\Gamma$: $\varphi < 0$ on $\partial\Omega \setminus \Gamma$.

PROOF. By using extension theorems we can find a function $u^*$ with the Cauchy data (3.2.5) such that the norms from the left side of (3.2.6) are bounded by $CF$. By subtracting this function from $u$, we may assume that its Cauchy data on $\Gamma$ are zero.

Let $\chi$ be a $C^\infty$ function that is 1 on $\Omega_{\varepsilon/2}$ and 0 near $\partial\Omega\setminus\Gamma$, $0 \le \chi \le 1$. Then $v$ defined as $\chi u$ is contained in $\mathring{H}_{(m_1)}(\Omega_0)$.

Using Leibniz's formula for the differentiation of the product we conclude that $A(\chi u) = \chi Au + A_{m-1}u$, where the last operator involves only $\partial^\alpha u$ with $|\alpha : m| < 1$. Observe that the bound (3.2.3) stated for infinitely smooth compactly supported functions by approximation can be extended onto $u \in \mathring{H}_{(m_1)}(\Omega)$. By applying to $v$ this estimate, we get

$$\tau \sum \|w\partial^\alpha v\|_2^2(\Omega_{\varepsilon/2}) \leq C\left(\|wf\|_2^2(\Omega) + \sum \|w\partial^\alpha u\|_2^2(\Omega)\right).$$

Since $v = u$ on $\Omega_{\varepsilon/2}$, we can replace $v$ by $u$, provided that we replace $\Omega$ by $\Omega_{\varepsilon/2}$ on the left side. By choosing $\tau > 2C$ and subtracting from both sides the integrals over $\Omega_{\varepsilon/2}$ multiplied by $C$, we can shrink $\Omega$ to $\Omega\backslash\Omega_{\varepsilon/2}$ on the right side. Since $\Omega_\varepsilon \subset \Omega_{\varepsilon/2}$, we obtain

$$\sum \|w\partial^\alpha u\|_2^2(\Omega_\varepsilon) \leq C(\|wf\|_2^2(\Omega) + \sum \|w\partial^\alpha u\|_2^2(\Omega\backslash\Omega_{\varepsilon/2})).$$

We have $\exp(\tau\varepsilon) < w$ on $\Omega_\varepsilon$, $w < \exp(\tau\Phi)$ where $\Phi$ is $\sup\varphi$ over $\Omega$, and $1 \leq w \leq \exp(\tau\varepsilon/2)$ on $\Omega\backslash\Omega_{\varepsilon/2}$. Replacing $w$ by its minimum on the left side and by its maximum over closures of the integration domains on the right side and dividing both sides by $\exp(2\tau\varepsilon)$, we get

$$\|\partial^\alpha u\|_2(\Omega_\varepsilon) \leq C\exp(\tau(\Phi-\varepsilon))F + C\exp(-\tau\varepsilon/2)M.$$

Let us choose

$$\tau = \max\{(\Phi - \varepsilon/2)^{-1}\ln(M/F), C(\varepsilon)\},$$

where $C(\varepsilon) > 0$ is needed to ensure that $\tau > 2C$. Due to this choice, the second term on the right side does not exceed the first one. After substituting the above expression $\tau$, we obtain (3.2.6) with $\lambda = \varepsilon/(2\Phi - \varepsilon)$.

The proof is complete. □

In many interesting situations the pseudo-convexity condition is not satisfied, but one can still prove uniqueness in the Cauchy problem. A classical example is Holmgren's theorem.

We recall that a surface $\Gamma$ given by the equation $\{\gamma(x) = 0, x \in \Omega\}$, $\gamma \in C^1(\Omega)$, is called noncharacteristic with respect to the operator $A$ if for the principal symbol $A_m(x;\zeta)(m_1 = \cdots = m_n)$ of this operator we have $A_m(x;\nabla\gamma(x)) \neq 0$ when $x \in \Gamma$.

**Theorem 3.2.3.** *Let us assume that the coefficients of the operator A are (real) analytic in a neighborhood of* $\overline{\Omega}$.

*If a surface* $\Gamma$ *is noncharacteristic with respect to A, then there is a neighborhood* $V$ *of* $\Gamma$ *such that a solution u to the Cauchy problem* (3.2.5) *is unique in* $\Omega_0 = \Omega \cap V$.

For a proof of this classical result we refer to the book of Fritz John [Jo4]. It is based on solvability of the noncharacteristic Cauchy problem for the adjoint operator in analytic classes of functions, on the density of such functions in $C^k$

and $H_{p,k}$ spaces, and on Green's formula. An immediate corollary is the following global version of this result.

Let $\Gamma_\tau$ be a family of noncharacteristic surfaces in a neighborhood of $\overline{\Omega}$ given by the equations $\{\gamma(x,t)=0\}$, $0 \leq \tau \leq 1$, where $\gamma \in C^1(Q)$ for some open set $Q$ in $\mathbb{R}^{n+1}$ containing $\Gamma_\tau \times \{\tau\}$. We assume that the boundary of $\Gamma_\tau$ does not intersect $\Omega$. We introduce $\Omega_{1-\tau} = \cup(\Gamma_\sigma \cap \Omega)$ over $0 < \sigma < \tau$.

**Corollary 3.2.4.** *Let us assume that $A$ has real-analytic coefficients in $\overline{\Omega}$ and the surfaces $\Gamma_\tau$, $0 \leq \tau \leq 1$, are noncharacteristic with respect to $A$. Let us assume that the $\Gamma_\tau$ do not intersect $\partial\Omega \backslash \Gamma$ and that $\Gamma_0 \cap \overline{\Omega} \subset \Gamma$.*

*Then a solution $u$ to the Cauchy problem* (3.2.5) *is unique in $\Omega_0$.*

This result follows from Theorem 3.2.3 by a standard compactness argument. Indeed, by this theorem $u = 0$ on $\Omega_{1-\varepsilon(0)}$, and if $u = 0$ on $\Omega_\tau$, then it is zero on $\Omega_{\tau-\varepsilon(\tau)}$ for some (small) positive $\varepsilon(\tau)$. The intervals $(\tau - \varepsilon(\tau), \tau)$ form an open covering of the compact interval $[\delta, 1-\delta]$ for any $\delta < \varepsilon(0)$. There is a finite subcovering $(\tau_1 - \varepsilon_1, \tau_1), \ldots, (\tau_k - \varepsilon_k, \tau_k)$ of this interval, and one can assume that $\tau_j - \varepsilon_j < \tau_{j+1} < \tau_j$. Moving from $j$ to $j+1$, we conclude that $u = 0$ on $\Omega_\delta$ for any $\delta$, and therefore on $\Omega_0$.

For a discussion of these classical results and for an explicit construction of the uniqueness domains $\Omega_0$ we refer to the books of Courant and Hilbert [CouH], pp. 238, and John [Jo4]. We observe that uniqueness domain in Corollary 3.2.4 is sharp, while uniqueness results of Theorem 3.2.2 are typically not sharp.

The assumption of analyticity of the coefficients is too restrictive for many applications, so the following result of Tataru [Tat2] is quite important.

Let $x = (x', x'')$ where $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$. We say that a function $\varphi \in C^2(\overline{\Omega})$ is strongly $'$ pseudo-convex with respect to the operator $A(m = (m_1, \ldots, m_1))$ if $\nabla\varphi(x) \neq 0$ and conditions (3.2.1) are satisfied for any $\xi = (\xi', 0)$ at any point $x \in \overline{\Omega}$.

**Theorem 3.2.5.** *Let us assume that $A$ is a differential operator with $x''$-independent coefficients. Let $\varphi$ be a strongly$'$ pseudo-convex function in $\Omega$, $\Omega_\epsilon = \Omega \cap \{\varepsilon < \varphi\}$, and $\overline{\Omega_0} \subset \Omega \cup \Gamma$.*

*Then a solution $u$ to the Cauchy problem (3.2.5) is unique in $\Omega_0$.*

In fact, Tataru proved a stronger result assuming analytic dependence of the coefficients on $x''$. A crucial idea of his proof is to apply to $u$ the pseudodifferential operator

$$e^{(\partial'')^2/\tau}, \tag{3.2.7}$$

which is the convolution with the Gaussian kernel in $x''$-variables

$$(\tau/(2\pi))^{(n-k)/2} e^{-\tau|x''-y''|^2/2},$$

while using Carleman estimates in $x'$. This idea brought weaker results, which appeared earlier in the paper of Robbiano [Ro]. In Section 3.4 we will show that

Theorem 3.2.5 gives a sharp description of the uniqueness domain in the lateral Cauchy problem for second-order hyperbolic equations with time-independent coefficients.

We observe that in the situations covered by Theorems 3.2.3, 3.2.5 one can generally expect only stability estimates of logarithmic type, and indeed, for a solution $u$ to the Cauchy problem (3.2.5) the estimates

$$\|u\|_2(\Omega_0) \leq C/|\ln F|,$$

where $C = C(M)$ and a priori $\|\partial^\alpha u\|_2(\Omega) \leq M$, $|\alpha : m| \leq 1$, have been obtained by Fritz John [Jo2] in the situation of Corollary 3.2.4. So Theorem 3.2.2 is very important due to the much better Hölder-type stability obtained.

Observe that Metivier [Me] found analytic nonlinear equations such that a noncharacteristic Cauchy problem has several smooth solutions. A surface is noncharacteristic at a solution of a nonlinear partial differential equation if is is noncharacteristic with respect to a linearization of this equation at this solution.

COUNTEREXAMPLE 3.2.6 (METIVIER [ME]). For the semilinear equation of a third order

$$(\partial_4 + \partial_3)(\partial_4^2 u + \partial_1^2 u - \partial_2^2 u + (\partial_4 u)^2 + (\partial_1 u)^2 - (\partial_2 u)^2) = 0$$

in $\mathbb{R}^4$ there are two different $C^\infty$-solutions in the neighborhood of the origin which coincide when $x_4 < 0$. Obviously, the surface $\{x_4 = 0\}$ is noncharacteristic at any smooth solution to this equation.

It is instructive to mention also the simple $2 \times 2$ system found in [Me]

$$\partial_3 u + v\partial_1 u = 0,$$

$$\partial_3 v - \partial_2 v = 0$$

in $\mathbb{R}^3$ with a similar property. Indeed, according to [Me] there are two different $C^\infty$-solutions $(u, v)$ to this system in a neighborhood of the origin in $\mathbb{R}^3$ which coincide when $x_3 < 0$. Again, the surface $\{x_3 = 0\}$ is noncharacteristic at any smooth solution to this system.

The Cauchy problem for some nonlinear (elliptic) partial differential equations is of applied importance. In particular, we mention the continuation of the wave field beyond caustics [MT].

## 3.3 Elliptic and parabolic equations

Now we derive from Theorem 3.2.2 and Theorem 3.5.2 (which is obtained independently on section 3.3) particular and more precise results for elliptic and parabolic equations of second order.

We consider the elliptic operator $Au = \sum a_{jk}\partial_j\partial_k u + \sum b_j\partial_j u + cu$ with real-valued $a_{jk} \in C^1(\overline{\Omega})$ and $b_j, c \in L_\infty(\Omega)$. Here the sums are over $j, k = 1, \ldots, n$.

**Theorem 3.3.1.** *For any domain* $\Omega_\varepsilon$ *with* $\overline{\Omega_\varepsilon} \subset \Omega \cup \Gamma$, *a solution u to the Cauchy problem* (3.2.5) *satisfies the following estimate:*

$$\|u\|_{(1)}(\Omega_\varepsilon) \le C(F + \|u\|_{(1)}^{1-\kappa}(\Omega) F^{\kappa}), \tag{3.3.1}$$

*where* $C$ *and* $\kappa \in (0, 1)$ *depend only on* $\Omega_\varepsilon$, *and* $F = \|f\|_2(\Omega) + \|g_0\|_{(1)}(\Gamma) + \|g_1\|_{(0)}(\Gamma)$.

PROOF. Let $x \in \Omega \cup \Gamma$. Then there is a function $\psi \in C^2(\overline{\Omega})$ with nonvanishing gradient in $\overline{\Omega}$ such that $0 < \psi(x)$ and $\psi < 0$ on $\partial\Omega \backslash \Gamma$. This function is pseudo-convex in $\overline{\Omega}$ with respect to $A$, hence by Theorem 3.5.2 we have the bound (3.3.1) where $\Omega_\varepsilon$ is replaced by a neighborhood $V(x)$ of $x$ in $\Omega$. $V(x)$, $x \in \overline{\Omega_\varepsilon}$ form an open covering of compact $\overline{\Omega_\varepsilon}$, so there is a finite subcovering $V_1, \dots, V_J$. By the choice,

$$\|u\|_{(1)}(V_j) \le C(F + \|u\|_{(1)}^{1-\kappa_j}(\Omega) F^{\kappa_j}) \le C(F + \|u\|_{(1)}^{1-\kappa}(\Omega) F^{\kappa})$$

where $\kappa = min(\kappa_1, \dots, \kappa_J)$. The last inequality can be easily obatined by considering the cases $F \le \|u\|_{(1)}(\Omega)$ and $\|u\|_{(1)}(\Omega) \le F$. From the definition of the $L_2$-norm we have

$$\|u\|_2(\Omega_\varepsilon) \le \|u\|_2(V_1) + \dots + \|u\|_2(V_J)$$

and using the above bounds in $V_j$ we complete the proof of (3.3.1). □

**Corollary 3.3.2.** *For any domain* $\Omega_\varepsilon$ *with* $\overline{\Omega_\varepsilon} \subset \Omega \cup \Gamma$, *a solution u to the Cauchy problem* (3.2.5) *satisfies the following estimate:*

$$\|u\|_{(1)}(\Omega_\varepsilon) \le C(F + \|u\|_2^{1-\kappa}(\Omega) F^{\kappa}), \tag{3.3.2}$$

*where* $C$ *and* $\kappa \in (0, 1)$ *depend only on* $\Omega_\varepsilon$, *and* $F = \|f\|_2(\Omega) + \|g_0\|_{(1)}(\Gamma) + \|g_1\|_{(0)}(\Gamma)$.

This corollary follows from Theorem 3.3.1 and known interior Schauder-type estimates for elliptic boundary value problems. Indeed, let $\Omega_0$ be a subdomain of $\Omega$ containing $\overline{\Omega_\varepsilon}$ with $\overline{\Omega_0} \subset \Omega \cup \Gamma$. From known interior Schauder-type estimates (Theorem 4.1) $\|u\|_{(1)}(\Omega_0) \le C(F + \|u\|_2(\Omega))$. Using this inequality in the bound (3.3.1) with $\Omega$ replaced by $\Omega_0$ we yield

$$\|u\|_{(1)}(\Omega_\varepsilon) \le C(F + (F + \|u\|_2(\Omega))^{1-\kappa} F^{\kappa}).$$

Now Corollary 3.3.2 follows by applying the inequality $(a + b)^{1-\kappa} \le a^{1-\kappa} + b^{1-\kappa}$, which is valid for all positive $a$, $b$.

Under additional regularity assumptions on $\Gamma$ one can similarly obtain (3.3.2) for $\|u\|_{(2)}(\Omega_\varepsilon)$ by using higher order interior Schauder-type estimates and (3.3.1).

In particular, Theorem 3.3.1 implies uniqueness of a solution to the Cauchy problem (3.2.5) for such equations: when $F = 0$, a solution $u$ is zero as well. Uniqueness is valid even under less restrictive assumptions that the coefficients $a_{jk}$ are Lipschitz (and not necessarily real-valued) (see the book of Hörmander [Hö2], section 17.2). An optimal assumption on $c(\in L_{n/2})$ is made in the paper of

Jerison and Kenig [JK], where they considered the stationary Schrödinger operator $\Delta u + cu = 0$.

These results show that an elliptic equation of second order with Lipschitz principal coefficients possesses the so-called *uniqueness of the continuation* property for their solutions: if a solution $u$ is zero on a subdomain $\Omega_0$ of a domain $\Omega$, then $u = 0$ on $\Omega$. To deduce it from Theorem 3.3.1, we introduce a ball $B$ with closure in $\Omega_0$. Then $u$ satisfies the homogeneous equation in the domain $\Omega\backslash\overline{B}$ and has zero Cauchy data on the part $\partial B$ of its boundary, so by Theorem 3.3.1 $u$ is zero in $\Omega\backslash B$.

We mention some important and surprising counterexamples.

COUNTEREXAMPLE 3.3.3 (Pliš [Pl2]). There is an elliptic equation

$$a_{11}(x_3)\partial_1^2 u + \partial_2^2 u + \partial_3^2 u + b_1(x)\partial_1 u + b_2(x)\partial_2 u + c(x)u = 0$$

with real-valued coefficients $a_{11} \in C^\lambda(\mathbb{R})$ for any $\lambda < 1, b_1, b_2, c \in C(\mathbb{R}^3)$, that has a $C^\infty(\mathbb{R}^3)$-solution $u$ such that $u = 0$ when $x_3 \leq 0$, but it is nonzero in any neighborhood of any point of $\{x_3 = 0\}$.

There are nonzero solutions of similar equations with compact supports.

Another interesting question is about the minimal size of the set $\Gamma$ where on prescribes the Cauchy data. In the two-dimensional case uniqueness and stability hold for $\Gamma$, which is closed and of positive measure on $\partial\Omega \in C^1$. We will show this by using the harmonic measure and some results of potential theory.

First we consider harmonic functions $u \in C^1(\Omega \cup \Gamma)$. We assume that $|\nabla u| < F$ on $\Gamma$ and $|\nabla u| < M$ on $\Omega$. We recall that the harmonic measure $\mu(x; \Gamma)$ of a closed set $\Gamma \subset \partial\Omega$ with respect to a point $x \in \Omega$ is a harmonic function of $x$ that can be defined as follows. Let a sequence of functions $g_k^+ \in C(\partial\Omega)$ be monotonically convergent to the characteristic function $\chi(\Gamma)$ of the set $\Gamma$, $g_k^+ \geq g_{k+1}^+$, and $\mu_k^+$ are harmonic functions with the boundary Dirichlet data $g_k^+$. Then the $\mu_k^+$ are monotonically convergent to $\mu$. We observe that if $meas_1\,\Gamma > 0$, then $\mu(x; \Gamma) > 0$ for any $x \in \Omega$. This follows from the standard representation of a solution $\mu_k^+$ to the Dirichlet problem via Green's kernel $G(x, y)$, which is positive and continuous when $x \in \Omega$, $y \in \partial\Omega$. We have

$$\mu_k^+(x) = \int_{\partial\Omega} G(x, y)g_k^+(y)d\Gamma(y) \geq \int_\Gamma G(x, y)d\Gamma(y) > 0$$

because $g_k^+ \geq 1$ on $\Gamma$.

The function $s(x) = \ln|\nabla u(x)|$ is a subharmonic function in $\Omega$. When $x \in \partial\Omega$, we have $\liminf(s(y) - (1 - \mu(y))\ln M - \mu(y)\ln F) \leq 0$ as $y \to x \in \partial\Omega$. Indeed, when $x \in \Gamma$, the left side is not greater than

$$\begin{aligned}&\liminf(\ln F - (1 - \mu(y)\ln M - \mu(y)\ln F)\\ &\leq \liminf(1 - \mu(y))(\ln F - \ln M)) \leq 0,\end{aligned}$$

because $\mu(y) \leq 1$ by the maximum principle, and $F \leq M$. When $x \in \partial\Omega\backslash\Gamma$, the function $\mu$ is continuous at $x$ and equal to 0 there, so the left side under consideration is equal to $\liminf(s(y) - \ln M) \le 0$. Since the function $s(x) - (1 - \mu(x))\ln M - \mu(x)\ln F$ is subharmonic and its upper limit on $\partial\Omega$ is $\le 0$ by the maximum principle, $s(x) \le (1 - \mu(x))\ln M - \mu(x)\ln F$ for all $x \in \Omega$. Taking exponents of both parts, we will have

$$|\nabla u(x)| \le M^{1-\mu(x)} F^{\mu(x)}.$$

Since $0 < \mu(x) < 1$, we have a conditional stability estimate that implies uniqueness (when $F \to 0$).

For recent results on minimal requirements on the uniqueness set $\Gamma$ we refer to the review paper of Aleksandrov, Bourgain, Gieseke, Havin, and Vymenetz [AlBGHV], where they give a particular answer to the question, for what $\Gamma$ do both uniqueness and existence for the Cauchy problem hold. As far as we know, even in $\mathbb{R}^2$ it is still an unresolved question.

In the three-dimensional case the assumption $meas_2\,\Gamma > 0$ is not sufficient. Recently, Wolff [Wo] (see also the paper of Bourgain and Wolff) [BoW] found a strong counterexample that disproves an old conjecture of Bers and M.A. Lavrentiev that this assumption is sufficient. We will formulate Wolff's counterexample.

COUNTEREXAMPLE 3.3.4. There is a nonzero $C^1(\overline{\mathbb{R}}^3_+)$-function, $u$ harmonic in the half-space $\mathbb{R}^3_+ = \{x_3 > 0\}$, and a closed subset $\Gamma \subset \partial\mathbb{R}^3_+$ of positive two-dimensional measure such that $u = |\nabla u| = 0$ on $\Gamma$.

Observe however, that a counterexample with $u \in C^2(\overline{\mathbb{R}}^3_+)$ is not known.

Returning to equations of second-order in the plane, we observe that any elliptic equation

$$(a_{11}\partial_1^2 + 2a_{12}\partial_1\partial_2 + a_{22}\partial_2^2 + b_1\partial_1 + b_2\partial_2 + c)u = 0 \tag{3.3.3}$$

with measurable and bounded coefficients $a_{jk}, b_j, c$ in a bounded plane domain $\Omega$ can be reduced to the particular case of these equations with $c = 0$ by the substitution $u = u^+ v$. Here $u^+ \in H_{2,p}(\Omega)$, $p > 1$, is a positive solution to the initial equation. The existence of $u^+$ can be shown for $\Omega$ of small volume by using standard elliptic theory. The equation with $c = 0$ can be reduced to an elliptic system for the vector function $w_1 = \partial_1 u, w_2 = \partial_2 u$. By the Bers-Nirenberg theory, any solution $w = w_1 + iw_2$ of such a system admits the representation $w(z) = e^{s(z)} f(\chi(z))$, where the functions $s, \chi$ are Hölder continuous on $\Omega$, $\chi$ is one-to-one there, and $f$ is a complex-analytic function of $z = x_1 + ix_2$ on $\chi(\Omega)$. Since the zeros of an analytic function $f$ are isolated, so are the zeros of $w = \nabla u$.

There is another useful concept for the uniqueness of the continuation. A point $a \in \Omega$ is called a zero of infinite order of a function $u \in L_1(\Omega)$ if for any natural number $N$ there is a constant $C(N)$ such that $\int_{B(a;r)} |u| \le C(N) r^N$. If $u \in C^\infty(\Omega)$, this definition is equivalent to the claim that all partial derivatives of $u$ are zero at $a$. From the above results it follows that if $a$ is a zero of infinite order for $u - u(a)$ for a solution $u$ to equation (3.3.3), then $u$ is constant in $\Omega$. Alessandrini observed that the same property holds for solutions to the important elliptic equation

$$\partial_1(a_{11}\partial_1 u + a_{12}\partial_2 u) + \partial_2(a_{21}\partial_1 u + a_{22}\partial_2 u) = 0.$$

Indeed, this equation is the integrability condition for the overdetermined system $\partial_1 v = -a_{21}\partial_1 u - a_{22}\partial_2 u$, $\partial_2 v = a_{11}\partial_1 u + a_{12}\partial_2 u$ with respect to $v$. So our equation implies the existence of a solution $v$ to this system. This system with respect to $u$ and $v$ is again elliptic and satisfies all the conditions of the Bers-Nirenberg representation theorem, and we can repeat the above argument. In particular, all zeros of gradients of nonconstant solutions to the equation $\operatorname{div}(a\nabla u) = 0$ are of finite order if $a$ is merely bounded and measurable.

For higher-order equations the situation is quite complicated. In fact, there are very convincing examples of nonuniqueness.

COUNTEREXAMPLE 3.3.5 (Pliš [Pl1]). There is a fourth-order elliptic equation

$$((\partial_1^2 + \partial_2^2 + \partial_3^2)^2 + x_3(\partial_1^2 + \partial_2^2)^2 - 1/2\partial_1^2 + b_1\partial_1 + c)u = 0$$

with $C^\infty(\mathbb{R}^3)$ coefficients $b_1, c$ and solution $u$ with $\operatorname{supp} u = \{0 \le x_3\}$. There are similar equations with compactly supported solutions. In the same paper there are examples of sixth-order elliptic equations with complex-valued smooth coefficients in the plane that moreover do not have the property of uniqueness of continuation.

On the other hand, in some important cases from elasticity theory one has uniqueness and stability.

**Exercise 3.3.6.** Let $A_1, A_2$ be second-order elliptic operators with $C^2(\overline{\Omega})$-coefficients. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and $\Gamma$ a $C^4$-smooth part of its boundary $\partial\Omega$. Show that an $H_{(4)}(\Omega)$-solution $u$ to the Cauchy problem

$$A_1 A_2 u = 0 \text{ in } \Omega, \qquad \partial_\nu^j u = g_j \text{ on } \Gamma,\ j = 0, \dots, 3,$$

for any subdomain $\Omega_\varepsilon \subset \overline{\Omega_\varepsilon} \subset \Omega \cup \Gamma$ satisfies the following estimate:

$$\|u\|_{(4)}(\Omega_\varepsilon) \le C(F + \|u\|_2^{1-\kappa}(\Omega)F^\kappa),$$

where $C, \kappa$ depend on $\Omega_\varepsilon$, $0 < \kappa < 1$, and $F = \|g_0\|_{(4)}(\Gamma) + \cdots + \|g_3\|_{(1)}(\Gamma)$.

We observe that Theorem 3.2.1 cannot be applied to the operator $A = \Delta\Delta$ because strong pseudo-convexity condition (3.2.2) is not satisfied for any function $\phi$. Indeed, the equality $A(\zeta) = 0$ implies that the left side of (3.2.2) is zero.

More general equations can be considered by using Carleman estimates with a second large parameter $\sigma$. In next result we consider $\psi(x) = |x - b|^2$ and we let $\Omega_\varepsilon = \Omega \cap \{\varepsilon < \psi\}$, $\varphi = e^{\sigma\psi}$.

**Theorem 3.3.7.** *For a second order elliptic operator A and a bounded domain* $\Omega$ *there are constants* $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon, \sigma)$ *such that*

$$\sigma \int_\Omega (\sigma\tau\varphi)^{3-2|\alpha|} e^{2\tau\varphi} |\partial^\alpha u|^2 \leq C_1 \int_\Omega e^{2\tau\varphi} |Au|^2$$

*for all* $u \in C_0^2(\Omega)$ *and* $|\alpha| \leq 2$ *provided* $C_1 < \sigma, C_2 < \tau$.

This result is obtained in [EII] by careful examining how constants in the proof in [Höl] depend on $\sigma$. Using Theorem 3.3.6 one can handle arbitrary third order perturbations of the product of two elliptic operators of second order.

**Exercise 3.3.8.** Let $A_1, A_2$ be second-order elliptic operators with $C^2(\overline{\Omega})$-coefficients. Let $A_3$ be a third-order linear partial differential operator with bounded and measurable coefficients in $\Omega$. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and $\Gamma$ a $C^4$-smooth part of its boundary $\partial\Omega$.

Show that an $H_{(4)}(\Omega)$-solution $u$ to the Cauchy problem

$$A_1 A_2 u + A_3 u = 0 \text{ in } \Omega, \qquad \partial_\nu^j u = g_j \text{ on } \Gamma, \; j = 0, \ldots, 3,$$

for any subdomain $\Omega_\varepsilon \subset \overline{\Omega_\varepsilon} \subset \Omega \cup \Gamma$ satisfies the following estimate:

$$\|u\|_{(4)}(\Omega_\varepsilon) \leq C(F + \|u\|_2^{1-\kappa}(\Omega) F^\kappa),$$

where $C, \kappa$ depend on $\Omega_\varepsilon$, $0 < \kappa < 1$, and $F = \|f\|_{(0)}(\Omega) + \|g_0\|_{(4)}(\Gamma) + \cdots + \|g_3\|_{(1)}(\Gamma)$.

To solve Exercise 3.3.8 we recommend first to cover $\overline{\Omega}_\varepsilon$ by a finite number of $C^4$-diffeomorphic images of the subset $\Omega_* = \{x : |x| < 1, x_n < -1/2\}$ of the unit ball such that the same diffeomorphic image of $\Gamma_* = \{x : |x| = 1, x_n < 0\}$ is contained in $\Gamma$. Since the form of the fourth order equation from Corollary 3.3.8 does not change under these diffeomorphisms, it suffices to assume that $\Omega_\varepsilon = \Omega_*$ and $\Gamma = \Gamma_*$. Then choose $b = (0, \ldots, 0, 1)$, apply Theorem 3.3.7 twice (to $A_1(A_2 u)$) to derive a Carleman estimate for $A_1 A_2$ with the second large parameter $\sigma$, and repeat the proof of Theorem 3.2.2 by using this new Carleman estimate.

COUNTEREXAMPLE 3.3.9. For fourth order equations prescribing three boundary conditions is not sufficient for uniqueness. By the Cauchy-Kovalevsky theorem for analytic $\Gamma$ and coefficients there many solutions near $\Gamma$. The following example gives global solutions in $\Omega$ when $\Gamma = \partial\Omega$.

Indeed, let $\Omega$ be the unit ball in $\mathbb{R}^3$, let $k_1^2, k_1 = \pi$ be the first (smallest) Dirichlet eigenvalue and $u_1$ be a corresponding eigenfunction

$$(\Delta + k_1^2) u_1 = 0, \qquad \text{in } \Omega, \; u_1 = 0 \qquad \text{on } \partial\Omega.$$

Observe that we can assume $u_1(x) = sin(\pi r)/r$, where $r = |x|$. Let $k_2 = 2\pi$ and

$$u(x) = \int_\Omega K(x - y) u_1(y) dy$$

where $K(x) = -e^{ik_2|x|}/(4\pi|x|)$ is the radiating fundamental solution to the Helmholtz operator $\Delta + k_2^2$. We have

$$(\Delta + k_2^2)u = \chi(\Omega)u_1 \qquad \text{in } \mathbb{R}^3.$$

Then $(\Delta + k_1^2)(\Delta + k_2^2)u = 0$ in $\Omega$, $u = \partial_\nu u = \partial_\nu^2 u = 0$ on $\partial\Omega$, but $u$ is not zero almost everywhere in $\Omega$.

To prove this statement it suffices to show that $u = \partial_\nu u = \partial_\nu^2 u = 0$ on $\partial\Omega$. Since $u_1 = 0$ on $\partial\Omega$, the function $\chi(\Omega)u_1 \in H_{2,p}(\mathbb{R}^3)$ for any $p > 1$. From known regularity properties of potentials, $u \in C^2(\mathbb{R}^3)$. Hence to complete the proof it is sufficient to show that $u(x) = 0$ when $|x| > 1$.

As known from the theory of the Helmholtz equation [CoKr], Theorem 2.10,

$$K(x-y) = \sum c_{n,m}(x) j_n(k_2|y|) Y_n^m(y/|y|)$$

where the sum is over $n = 0, 1, 2, \ldots, m = -n, \ldots, n$, the $j_n$ is the spherical Bessel function and $Y_n^m$ are standard spherical harmonics, and the series is uniformly convergent on $\Omega$ when $|x| > 1$. Combining this series representation with the integral formula for $u$, we complete the proof if we show that

$$\int_\Omega u_1(y) j_n(2\pi|y|) Y_n^m(y/|y|) dy = 0$$

for all $n = 0, 1, 2, \ldots, m = -n, \ldots, n$. Since $u_1$, $j_n(2\pi|y|)$ do not depend on spherical angles, by using polar coordinates this equality follows from basic orthogonality property of spherical harmonics when $n = 1, 2, \ldots$. When $n = 0$, $Y_0^0$ is constant and by using again polar coordinates and the formulas $u_1(y) = sin(\pi r)/r$, $j_0(r) = sinr/r$, the needed equality is reduced to

$$\int_0^1 sin(\pi r) sin(2\pi r) dr = 0$$

which follows by elementary integration.

In the remaining part of this section we consider the second-order parabolic operator $Au = \partial_t u + A'u$, where $A'$ is the second-order elliptic operator considered above with coefficients $a_{jk} \in C^1(\overline{\Omega})$; $b_j, c \in L_\infty(\Omega)$. We let $t = x_{n+1}$ and choose $m = (2, \ldots, 2, 1)$.

**Theorem 3.3.10.** *Let $\Omega = G \times I$ and $\Gamma = \gamma \times I$, where $G$ is a domain in $\mathbb{R}^n$, $\gamma$ is a $C^2$-smooth part of its boundary, and $I = (0, T)$.*

*Then for any domain $\Omega_\epsilon$ with closure in $\Omega \cup \Gamma$, a solution $u$ to the Cauchy problem* (3.2.5) *satisfies the following estimate:*

$$\|\partial^\alpha u\|_2(\Omega_\varepsilon) \le C(F + \|u\|_2^{1-\kappa}(\Omega)F^\kappa) \qquad \text{when } |\alpha : m| \le 1, \tag{3.3.4}$$

*where $C$ and $\kappa \in (0, 1)$ depend on $\Omega_\varepsilon$, and $F$ is given in Theorem* 3.3.1.

PROOF. We first consider $\Omega$, which is the half-ball $\{|x| < 1, x_n < 0\}$ in $\mathbb{R}^{n+1}$. We will make use of the weight function $\varphi(x) = exp(-\sigma x_n)$. Then

$\zeta = (\xi_1, \ldots, \xi_{n-1}, \xi_n - i\tau\sigma \exp(-\sigma x_n),\ \xi_{n+1})$. The equality $A_m(x; \zeta) = 0$ is equivalent to the equalities

$$\sum a_{jk}\xi_j\xi_k = a_{nn}\tau^2\sigma^2 exp(-2\sigma x_n),\ \xi_{n+1} - 2\sum a_{jn}\xi_j\tau\sigma \exp(-\sigma x_n) = 0. \tag{3.3.5}$$

The left side of (3.2.1) is

$$\sigma^2 \exp(-\sigma x_n)\left(4\left(\sum a_{jn}\xi_j\right)^2 + 4(a_{nn})^2\tau^2\sigma^2 \exp(-2\sigma x_n)\right)$$
$$+ 2/\tau\Im\sum(\partial_k a_{jl})a_{kq}\zeta_j\zeta_l\overline{\zeta}_q.$$

Standard calculations show that the last sum consists of terms $\sigma b\xi_j\xi_l exp(-2\sigma x_n)$, $\sigma^3 b\tau^2 exp(-3\sigma x_n)$ where $b$ are bounded functions. The first equality (3.3.5) and ellipticity of $A'$ imply that $|\xi| \leq C\tau\sigma exp(-\sigma x_n)$. Observing powers of $\sigma$ and choosing $\sigma$ large we achieve positivity of the left side of (3.2.2). We can assume that $\Omega_\epsilon$ is $\{|x| < 1, x_n < -\varepsilon\}$. Now from Theorem 3.2.2 we have the bound (3.3.4) with $M$ instead of $\|u\|_2(\Omega)$. To obtain (3.3.4) from this bound we can use the known Schauder-type estimates for second-order parabolic boundary value problems [LSU] $\|\partial^\alpha u\|_2(\Omega_{\varepsilon/2}) \leq C(F + \|u\|_2(\Omega_{\varepsilon/4}))$ when $|\alpha : m| \leq 1$ and argue as in the proof of Corollary 3.3.2.

By using the substitution $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}, y_n = x_n + g(x_1, \ldots, x_{n-1}, x_{n+1}),\ y_{n+1} = x_{n+1}$ making $\Gamma$ parallel to the $t$-axis, one reduces any cylindrical domain $B_- \times I$ ($B$ is the unit half-ball in $\mathbb{R}^n$) to the half-ball in $\mathbb{R}^{n+1}$. Observe that parabolicity of the equation is preserved under this substitution. To reduce $G$ to $B_-$, one can argue now as in the elliptic case.

The proof is complete. □

As in the elliptic case, a minor reduction of the regularity assumptions on the coefficients is possible. In particular, we mention the paper of Knabner and Vessella [KnV], in which they consider the one-dimensional case ($n = 1$) and obtain stability estimates assuming only that $a_{11}$, $\partial_1 a_{11}$ are continuous. Similar results are unknown in higher dimensions.

Also, as for elliptic equations, we can claim that under these regularity assumptions on the coefficients we have the following lateral uniqueness continuation property: if $u = 0$ on $G_0 \times I$ for some nonempty open subset $G_0$ of $G$, then $u = 0$ on $\Omega$.

An interesting consequence of Theorem 3.3.10 and of $t$-analyticity of solutions of parabolic boundary value problems with $t$-independent coefficients is the following result generalizing the backward uniqueness property.

**Corollary 3.3.11 (Local backward uniqueness).** *Let $u$ be a solution to the evolution equation* (3.1.1)*, where $A$ is an elliptic partial differential operator of second order in $G$ (with $t$-independent coefficients) satisfying the regularity assumptions of Theorem 3.3.10. The domain of $A$ is $H^0_{(1)}(G) \cap H_{(2)}(G)$.*

*If $u = 0$ on $G_0 \times \{T\}$, where $G_0$ is a nonempty open subset of $G$, then $u = 0$ on $\Omega$.*

PROOF. It is known that a solution $u(x, t)$ of a parabolic boundary problem with zero boundary data and time-independent coefficients is analytic with respect to $t \in (0, T)$ for any $x \in G$. Since $u = 0$ on $G_0 \times \{T\}$, we have $\partial_t u = -Au = 0$ on this set. Since the coefficients of $A$ are $t$-independent, $\partial_t u$ satisfies the same equation and lateral boundary conditions, so we can repeat the argument and conclude that all $t$-derivatives of $u$ are zero at $(x, T)$ when $x \in G_0$. Due to analyticity, $u = 0$ on $G_0 \times I$. Then $u = 0$ on $\Omega$ by the lateral uniqueness of continuation guaranteed by Theorem 3.3.10. □

This corollary is a simple consequence of $t$-analyticity of solutions of parabolic boundary value problems and of Theorem 3.3.10 on uniqueness of continuation for parabolic equations. On the other hand, quite recently Alessandrini, Escauriaza, Fernandez, and Vessella [AlVe], [F] obtained a uniqueness result that justifies the uniqueness of the continuation for solutions of a wide class of parabolic equations of second order under minimal assumptions. They proved that if a solution $u(x, t)$ of such a parabolic equation in the divergent form (with $C^1(\overline{\Omega})$-principal and $L_\infty(\Omega)$ other (time-dependent) coefficients) is 0 on $G_0 \times \{T\}$ for some nonempty open subset of $G$, then $u = 0$ on $G \times \{T\}$.

As for elliptic equations there is a Carleman type estimate with a second large parameter $\sigma$ where we let $\psi(x) = |x - b|^2$, $\varphi = e^{\sigma\psi}$.

**Theorem 3.3.12.** *For a second order parabolic operator $A$ and a bounded domain $\Omega$ there are constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon, \sigma)$ such that*

$$\sigma \int_\Omega (\sigma\tau\varphi)^{3-2|\alpha|} e^{2\tau\varphi} |\partial^\alpha u|^2 \leq C_1 \int_\Omega e^{2\tau\varphi} |Au|^2$$

*for all $u \in C_0^2(\Omega)$ and $|\alpha| \leq 2$, $\alpha_{n+1} \leq 1$ provided $C_1 < \sigma$, $C_2 < \tau$.*

This result is obtained by Eller and Isakov [ElI], and in section 3.5 we show how to use it to study the thermoelasticity system.

## 3.4 Hyperbolic and Schrödinger equations

In this section we consider the most challenging hyperbolic equations, where the uniqueness for the lateral Cauchy problem is much less understood.

Unless explicitly mentioned, in this section we consider the isotropic wave operator

$$A(x, t; \partial)u = a_0^2 \partial_t^2 u - \Delta u + \sum b_j \partial_j u + cu,$$

where $t = x_{n+1}$ and the sum is over $j = 1, \dots, n+1$, in the cylindrical domain $\Omega = G \times (-T, T)$ in $\mathbb{R}^{n+1}$. We assume that $a_0 \in C^1(\overline{\Omega})$, $a_0 > 0$, $b_j, c \in L_\infty(\Omega)$.

We will use the following weight function:

$$\varphi(x,t) = \exp((\sigma/2)\psi(x,t)) \tag{3.4.1}$$

with the two choices of $\psi$:

$$\psi_1(x,t) = x_1^2 + \cdots + x_{n-1}^2 + (x_n - \beta_n)^2 - \theta^2 t^2 - s$$

or

$$\psi_2(x,t) = -x_1^2 - \cdots - x_{n-1}^2 - \theta_n(x_n - \beta_n)^2 - \theta^2 t^2 + r^2 + \theta_n \beta_n^2.$$

In contrast to Section 3.2, we set $\Omega_\varepsilon = \Omega \cap \{\psi > \varepsilon\}$. We let $x' = (x_1, \dots, x_{n-1}, 0, 0)$.

**Theorem 3.4.1.** *Let $G$ be a domain in $\mathbb{R}^n$ and $\gamma \in C^2$ be a part of its boundary such that one of the following three conditions is satisfied:*

$$\textit{The origin belongs to } G;\ \gamma = \partial G. \tag{3.4.2}$$

$$G \textit{ is a subset of } \{-h < x_n < 0, |x'| < r\};\ \gamma = \partial G \cap \{x_n < 0\}. \tag{3.4.3}$$

$$G \textit{ is } \{-h < x_n < 0, |x'| < r\}; \gamma = \partial G \cap \{x_n = 0\}. \tag{3.4.4}$$

*Let us suppose that in the cases* (3.4.2) *and* (3.4.3)

$$\theta^2 a_0^2(a_0 + t\partial_t a_0 + 2a_0^{-1}|t\nabla a_0|) < a_0 + x \cdot \nabla a_0 - \beta_n \partial_n a_0, \theta a_0 \le 1, \tag{3.4.5}$$

*and in addition, $G \subset B(0;\theta T), \beta = s = 0$ in case* (3.4.2) *and $h(h + 2\beta_n) < \theta^2 T^2, s = \beta_n^2 + r^2$ in case* (3.4.3). *In case* (3.4.4) *we suppose that*

$$a_0 + \nabla' a_0 \cdot x + \theta_n \partial_n a_0 x_n + \delta_0 < \beta_n \theta_n \partial_n a_0,$$

$$\theta^2(a_0^3 + 2a_0|t\nabla a_0| - a_0^2 t\partial_t a_0) < \delta_0, a_0\theta r < \beta_n \theta_n^2, \quad r < \theta T \quad \textit{on } \overline{\Omega}. \tag{3.4.6}$$

*Let $\psi$ be $\psi_1$ in cases* (3.4.2) *and* (3.4.3) *and $\psi_2$ in case* (3.4.4).

*Then a solution $u$ to the Cauchy problem* (3.2.5) *admits the following bound:*

$$\|u\|_{(1)}(\Omega_\varepsilon) \le C(F + \|u\|_{(1)}(\Omega)^{1-\kappa} F^\kappa)$$

*where $C, \kappa \in (0,1)$ depend on $\varepsilon$, and $F = \|f\|_2(\Omega) + \|g_0\|_{(1)}(\Gamma) + \|g_1\|_{(0)}(\Gamma)$, $\Gamma = \gamma \times (-T, T)$.*

We illustrate case (3.4.2) in Figure 3.2 and case (3.4.3) in Figure 3.3.

PROOF. Let us consider cases (3.4.2) and (3.4.3). We let $\beta = (0, \dots 0, \beta_n)$. By standard calculations we obtain that the left side of (3.2.2′) is

$$\begin{gathered}
8\xi_1^2 + \cdots + 8\xi_n^2 - 8\theta^2 a_0^4 \xi_{n+1}^2 \\
-8a_0^3 \partial_t a_0 t\theta^2 \xi_{n+1}^2 + 16\sum a_0 \partial_k a_0 \xi_{n+1}\xi_k \theta^2 t \\
+8\sum a_0 \partial_j a_0 (x_j - \beta_j)\xi_{n+1}^2 = 8(1 + a_0^{-1}\nabla a_0 \cdot (x - \beta) \\
+\theta^2(-a_0^2 - a_0 t\partial_t a_0 + 2a_0 t\nabla a_0 \cdot \xi\xi_{n+1}) \\
\ge 8(1 + a_0^{-1}\nabla a_0 \cdot (x - \beta) - \theta^2(a_0^2 + a_0 t\partial_t a_0 + 2|t\nabla a_0|))
\end{gathered}$$

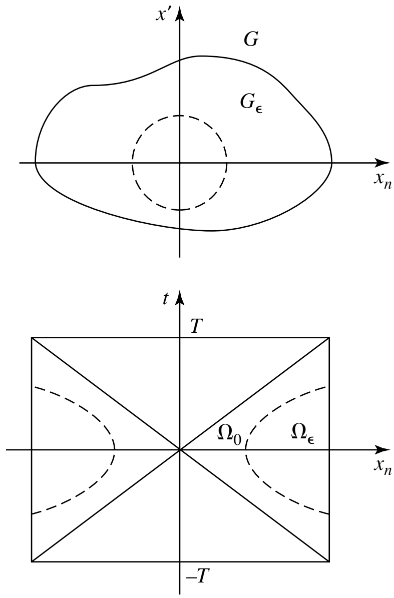


FIGURE 3.2.

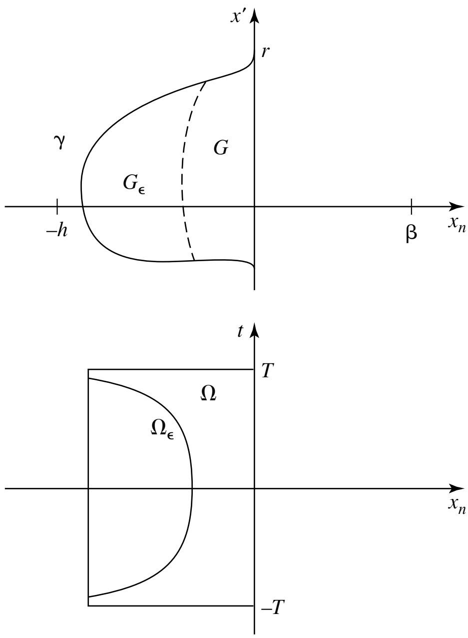


FIGURE 3.3.

where by homogeneity reasons we let $\xi_1^2 + \cdots + \xi_n^2 = 1$, used that then for a characteristic $\xi$ we have $a_0^2 \xi_{n+1}^2 = 1$ and applied the Schwarz inequality. So the first condition (3.4.5) guarantees the inequality (3.2.2′). The second condition (3.4.5) implies that $\nabla\psi$ is noncharacteristic on $\overline{\Omega}_0$. Hence the inequalities (3.4.5) guarantee pseudo-convexity of $\psi$.

Now we consider case (3.4.4). As above, the left side in (3.2.2′) is

$$\begin{aligned}
&-8(\xi_1^2 + \cdots + \xi_{n-1}^2 + \theta_n^2\xi_n^2) - 8\theta^2 a_0^4 \xi_{n+1}^2 + 16\theta^2 t a_0 \xi_{n+1} \nabla a_0 \cdot \xi \\
&-16\theta^2 a_0^3 t \partial_t a_0 \xi_{n+1}^2 - 8a_0 \nabla' a_0 \cdot x \xi_{n+1}^2 - 8\theta_n a_0 \partial_n a_0 x_n \xi_{n+1}^2 + 8\theta_n a_0 \partial_n a_0 \beta \xi_{n+1}^2 \\
&= -8 + 8(1-\theta_n^2)\xi_n^2 - 8\theta^2 a_0^2 - 16\theta^2 t a_0 \xi_{n+1} \nabla a_0 \cdot \xi - 16\theta^2 a_0 t \partial_t a_0 8 a_0^{-1} \nabla' a_0 \cdot x \\
&\quad - 8\theta_n a_0^{-1} \partial_n a_0 x_n + 8\theta_n a_0^{-1} \partial_n a_0 \beta + 8\theta^2 a_0 t \partial_t a_0 \\
&\geq 8\theta_n a_0^{-1} \partial_n a_0 \beta_n - 8 - 8a_0^{-1} \nabla' a_0 \cdot x - 8\theta_n a_0^{-1} \partial_n a_0 x_n \\
&\quad - 8\theta^2 (a_0^2 + 2|t\nabla a_0| - 8 a_0 t \partial_t a_0),
\end{aligned}$$

where we assumed that $\xi_1^2 + \cdots + \xi_n^2 = 1$ and used that due to the condition $A(,\xi) = 0$ we have $a_0^2 \xi_{n+1}^2 = 1$. The first two conditions (3.4.6) imply that the left side in (3.2.2′) is positive. The third inequality (3.4.6) implies that $\nabla\psi$ is noncharacteristic, and the fourth condition guarantees that $\psi(, T) < 0$ on $G$.

More detail in case (3.4.4) is given in [Is2].

In all cases, the conditions of Theorem 3.5.2 is satisfied, and Theorem 3.4.1 follows. □

Now we discuss conditions (3.4.5) and (3.4.6). Condition (3.4.5) is required for pseudo-convexity. If $a_0 + x \cdot \nabla a_0$ is positive, it can achieved by choosing $\theta$ to be small and $\beta_n = 0$. If this quantity is arbitrary but $\partial_n a_0 < 0$, we can guarantee (3.4.5) when $\beta_n$ is large. The condition $G \subset B(0;\theta T)$ is needed to ensure that $\psi < 0$ on $\partial\Omega \setminus \Gamma$. The same role is played by the condition $h(h + 2\beta_n) < \theta^2 T^2$ in case (3.4.3) and by the conditions $\theta = r/T$, $r^2 < 2h(h + \beta_n)$, which is easy to discover by elementary analytic geometry. This means that the observation time must be sufficiently large.

In case (3.4.2) with constant $a_0$, condition (3.4.5) is satisfied for any $\theta < 1/a_0$, and we obtain a sharp description of the uniqueness region $\Omega_0$ in the Cauchy problem with the data on the whole lateral surface $\partial G \times (-T, T)$ when $G$ is the ball $B(0;\theta T)$. In cases (3.4.3) and (3.4.4), the descriptions of the uniqueness region are not sharp. It is interesting to obtain more information about these regions. In case (3.4.3), the minimum $x_*$ of $x_n$ on $\partial G_0$ can be found from the equation $(x_* - \beta_n)^2 = \beta_n^2 + r^2$. It is clear that $G \cap \Omega_0 \cap \{t = 0\}$ contains $G \cap \{x_n < x_*\}$. We have

$$x_* = -\beta_n((1 + (r/\beta_n)^2)^{1/2} - 1) \geq -r^2/(2\beta_n).$$

So, by choosing $\beta_n$ large, we can guarantee uniqueness and stability in $G \cap \{x_n < x_*\}$ for any negative $x_*$.

Now we will comment on a choice of $\beta, \theta, T$ to satisfy conditions (3.4.5) and the inequalities $h(h + 2\beta_n) < \theta^2 T^2$. We first choose $\beta_n$ so that $0 < a_0 + x \cdot \nabla a_0 -$

$\beta_n \partial_n a_0$. It is possible if $0 < a_0 + x \cdot \nabla a_0$, in addition, when $\partial_n a_0 \le 0$, $\beta_n$ be arbitrarily large. Then we choose (large) $T$ so that

$$h(h+\beta_n)/T^2 < (a_0 + x \cdot a_0 - \beta_n \partial_n a_0)(a_0^3 + a_0^2 t \partial_t a + a_0 |t \nabla a_0|)^{-1}.$$

After that we let $\theta^2$ to be any number in between the left and the right side of the last inequality. Finally, it suffices for $a_0$ has to be monotone in some direction, and $T$ needs to be large. In more detail case (3.4.4) is discussed in [Is2].

For general (anisotropic) hyperbolic operator

$$A(x,t;\partial) = \partial_t^2 u - \sum a_{jk}(x)\partial_j \partial_k u + \sum b_j \partial_j u + cu$$

with $C^1$-coefficients the natural canididate for pseudo-convex function is

$$\varphi(x,t) = d(x) - \theta t^2$$

where $d$ is strictly convex in the Riemannian metric $\sum a^{jk}(x) dx_j dx_k$ ($a^{jk}$ is the inverse of $a_{jk}$) and $\nabla d \neq 0$ on $\overline{G}$. Indeed, Lasiecka, Triggiani, and Yao [LaTY] and Triggiani and Yao [TrY] obtained Carleman type estimates similar to (3.2.3′) with this weight function and with semi-explicit boundary terms. They used tools from Riemannian geometry and some pointwise inequalities for differential quadratic forms which trace back to theory of energy estimates for general hyperbolic equations and which are in simplest case outlined after Exercise 3.4.5. Of course, it is a problem to construct such $d$ for a particular choice of $a_{jk}$, $G$, $\Gamma$, but it can be done in some intersting cases.

COUNTEREXAMPLE 3.4.2. Fritz John [Jo2] showed that the Hölder-type stability on compact subsets stated in Theorem 3.4.1 is impossible when one considers the Cauchy problem for the wave operator $Au = \partial_t^2 u - \Delta u$ in $\Omega = G \times (-T, T)$ with $G = \{x : |x| > 1\}$ in $\mathbb{R}^2$. John found that the solutions to the wave equation

$$u_k(x,t) = k^{1/3} j_k(kr) e^{ik(t+\phi)},$$

($x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $j_k$ is the Bessel function of order $k$) satisfy the following conditions:

$$\|u_k\|_\infty(\Omega_{1/2}) \le q^k,\ 1/C \le \|u_k\|_\infty(\Omega_1),\ |u_k|_0(\mathbb{R}^3) \le C,$$

provided $C < k$ where $\Omega_{1/2}$ is the cylinder $\{|x| < 1/2, t \in \mathbb{R}\}$, $\Omega_1$ is the cylinder $\{|x| < 1, t \in \mathbb{R}\}$, and $q$, $C$ are constants independent of $k$, $0 < q < 1$. These solutions do not satisfy the estimate (3.2.6) (with any $\kappa \in (0,1)$). In fact, they show that the best possible estimate is of logarithmic type. This estimate is obtained in the same paper of John. Using the known recursion formulae $2j_k' = j_{k+1} + j_{k-1}$ one can replace the upper bound on $u_k$ in $C(\mathbb{R}^3)$ by a similar upper bound in any $C^k(\mathbb{R}^3)$.

Since the functions

$$v_k(x) = k^{1/3} j_k(kr) e^{ik\phi}$$

satisfy the Helmholtz equation $(\Delta + k^2)v_k = 0$ in $\mathbb{R}^2$ and $v_k(x) = e^{-ikt} u_k(x,t)$ the John's example also shows that the bound (3.2.6) or the conditional Hölder-type stability with constants independent on $k$ for general geometry is not possible for the Cauchy problem for the Helmholtz equation.

By using John's construction, Kumano-Go [KuG] found the operator $Au = \partial_t^2 u - \Delta u + b\partial_t u + cu = 0$ with time-dependent, real-valued $C^\infty(\mathbb{R}^3)$-coefficients $b, c$ such that the equation $Au = 0$ has a $C^\infty$-solution $u$ in $\mathbb{R}^3$ that is zero for $|x| < 1$ and not identically zero. So we have very convincing examples showing that conditions (3.4.5) and (3.4.6) are essential.

Considering $x$ and $y$ as elements of the field $\mathbb{C}$, introducing the conformal substitution $x = 1/(1-y) - 1/2$, and using the equality $\Delta_y = |x + 1/2|^4 \Delta_x$, we transform Kumano-Go's equation into our hyperbolic equation with $a_0 = 1/(x_1^2 + (x_2 + 1/2)^2)$ having a nonzero solution near the origin with zero Cauchy data on the plane $x_2 = 0$. Here $\partial_2 a_0 < 0$, so condition (3.4.6) is not satisfied.

If one allows complex-valued coefficients, then there are even counterexamples when the principal part has constant coefficients and $\Gamma = \{x_2 = 0\}$. If in Corollary 13.6.7 of Hörmander's book [Hö2] one sets the operator $Q(x, t; \partial) = \partial_1^2 + \partial_2^2 - \partial_t^2$, $\psi = x_2 + t$, and $\phi = x_1$, then one gets complex-valued $C^\infty$-functions $u(x, t), c(t + x_2, x_1)$ such that $(\partial_t^2 - \partial_1^2 + c(\partial_2 + \partial_t))u = 0$ on $\mathbb{R}^3$ and $\operatorname{supp} u = \{0 \le x_1\}$.

The following result is obtained in [ElI], [Is1] and it can be used when studying higher order equations and systems.

**Theorem 3.4.3.** *Let $a_0\theta < 1$ on $\overline{\Omega}$ and the condition (3.4.5) be satisfied.*

*Then there are constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon, \sigma)$ such that*

$$\int_\Omega (\sigma\tau\varphi)^{3-|\alpha|} e^{2\tau\alpha} |\partial^\alpha u|^2 \le C_1 \int_\Omega e^{2\tau\varphi} |Au|^2$$

*for all $u \in C_0^2(\Omega)$ provided $|\alpha| \le 1$, $C_1 < \sigma$, $C_2 < \tau$.*

Conditional Hölder and moreover logarithmic stability are reasons of poor resolution in numerical solution of many inverse problems which severely restricts applications. So better stability is very valuable in inverse problems. The above example of John shows that in a general case in the Cauchy problem for the Helmholtz equation stability is deteriorating when wave frequency $k$ grows. However, under natural convexity conditions the opposite is true, i.e. stability is improving. We will give one of the first results in this direction obtained by Hrycak and Isakov [HrI].

**Theorem 3.4.4.** *Let $\Omega(d) = \Omega \cap \{d < x_n\}$ and $\Gamma = \partial\Omega \cap \{d < x_n\}$.*

*Then there are constants $C$ and $\kappa \in (0, 1)$, $\kappa = \kappa(d)$, such that for any solution $u$ to the Cauchy problem (3.2.5) with $A = -\Delta - k^2$ we have*

$$\|u\|_{(0)}(\Omega(d)) \le C(F + (k+1)^{-1} d^{-2} \|u\|_{(1)}^{1-\kappa}(\Omega) F(k)^\kappa)$$

*where*

$$F(k) = \|f\|_{(0)}(\Omega) + (kd^{-0.5} + d^{-1.5})\|u\|_{(0)}(\Gamma) + \|\nabla u\|_{(0)}(\Gamma).$$

This result is proven in [HI] by splitting $u$ into a "low" and "high" frequency zones in the horizontal directions $x'$ by using the Fourier transformation with respect to $x'$. The "low" frequency component where $|\xi'| < k$ solves the hyperbolic differential equation with respect to $x_n$ and the Lipschitz stability bound $CF$ can be obtained by using standard energy estimates. To handle the "high" frequency component one observes that by elementary properties of the Fourier transformation the $H_{(0)}$-norm of this component is bounded by $(k+1)^{-1}$ times $H_{(1)}$-norm. To bound this higher order norm one can use Theorem 3.2.2 for $A$ with $C$ which does not depend on $k$. This $k$-independent bound can be obtained by repeating the proof of Theorem 3.2.2 where instead of Theorem 3.2.1 (or 3.2.1′) one uses the following Carleman estimate.

**Exercise 3.4.5.** Let $\varphi(x) = |x - \beta|^2$ and $\Omega$ be a bounded Lipschitz domain.

Show that there is a constant $C$ which does not depend on $k$ such that

$$\int_\Omega e^{2\tau\varphi}(\tau^3|u|^2 + \tau|\nabla u|^2) \le C\Big(\int_\Omega e^{2\tau\varphi}|(\Delta + k^2)u|^2 + \int_{\partial\Omega} e^{2\tau\varphi}(\tau^3|u|^2 + \tau|\nabla u|^2)\Big)$$

for $C < \tau$, all real $k$, and all $u \in H_{(2)}(\Omega)$.

To solve exercise 3.4.5 we recommend to use the basic ideas from theory of Carleman estimates. First, remove the exponential weight by using the substitution $u = e^{-\tau\varphi}v$. Then $A(, \partial)$ will be replaced by $A(, \partial - \tau\nabla\varphi)$. To get the $L_2$-bound for $v$ use the obvious inequality

$$|A(, \partial - \tau\nabla\varphi)v|^2 \ge |A(, \partial - \tau\nabla\varphi)v|^2 - |A(, \partial + \tau\nabla\varphi)v|^2.$$

Standard calculations show that for $A = -\Delta - k^2$ and $\varphi(x) = |x - \beta|^2$ the last expression is

$$-16\tau(\Delta v)(x - \beta)\cdot\nabla v - 8\tau n v\Delta v$$

$$-16\tau(4\tau^2|x-\beta|^2 + k^2)v(x-\beta)\cdot\nabla v - 8\tau n(4\tau^2|x-\beta|^2 + k^2)v^2.$$

One can complete the proof integrating by parts terms involving $\Delta v\partial_j v$, $v\Delta v$, $v\partial_j v$ and using that $v\partial_j v = 1/2\partial_j(v^2)$.

Now we give a sharp uniqueness result for equations with time-independent coefficients that will be derived from Theorem 3.2.5.

**Corollary 3.4.6.** *Let the coefficients of the operator $A$ be independent of $t$, let the coefficients of the principal part $a_{jk} \in C^1(\mathbb{R}^n)$, and let other coefficients be in $L_\infty(\mathbb{R}^n)$. Let $\Gamma \subset \Omega$ be a $C^2$-surface that is noncharacteristic with respect to $A$. Let us assume that $u \in H_{(1)}(\Omega)$, $Au = 0$ in $\Omega$.*

*If $u = \partial_\nu u = 0$ on $\Gamma$, then $u = 0$ in $\Omega$ near $\Gamma$.*

PROOF. Since uniqueness in the Cauchy problem with the data on a space-like surface is well known, we will assume that $\Gamma$ is time-like. Let $(x^0, t^0)$ be any point of $\Gamma$ and let $\Gamma$ be given by the equation $\{\psi(x,t) = 0\}$ near this point. We check the strong $'$-pseudo-convexity of the function $\varphi = \exp(\sigma\psi)$ for large $\sigma$, where we let $x' = x$ and $x'' = t$. We can assume that $|\zeta| = 1$. When $\tau = 0$, this condition is satisfied because at $\xi = \xi'$ the symbol is elliptic. By continuity it is satisfied also for $|\tau| < \varepsilon_0$. Let $|\tau| \geq \varepsilon_0$. Calculating as above, we conclude that the form (3.2.1) is $\mathfrak{H}_1 + \mathfrak{H}_2$, where $\mathfrak{H}_1 \geq -C\sigma\phi$ and

$$\mathfrak{H}_2 \geq \sigma^2\phi^3\tau^2|\partial_1\psi\partial_1\psi + \cdots + \partial_n\psi\partial_n\psi - a_0^2\partial_t\psi\partial_t\psi| \geq \sigma^2\phi/C$$

because the surface $\Gamma$ is noncharacteristic and because from $|\zeta| = 1$, $|\tau| \geq \varepsilon_0$ we have $|\tau\sigma\phi| > 1/C$. By choosing $\sigma$ large, we obtain the $'$-pseudo-convexity, which will be preserved for the small quadratic perturbation $\phi_{\varepsilon,r} = \varphi - \varphi(x^0, t^0) - \varepsilon(\rho^2 - r^2)$ of this function, where $\rho$ is the distance to the point $(x^0, t^0)$. The perturbed function is equal to $\varepsilon r^2 > 0$ at the point $(x^0, t^0)$, so it does satisfy the conditions of Theorem 3.2.5. So $u = 0$ on the set $\{\phi_{\varepsilon,r} > 0\} \cap \Omega$ (on one side of $\Gamma$ near the point $(x^0, t^0)$). If needed, we similarly consider the other side. Since this is an arbitrary point of $\Gamma$, the proof is complete. □

**Lemma 3.4.7.** *Let $Tr$ be the triangle in the $(x_n, t)$-plane with vertices* $(0, 0)$, $(-R, T)$, $(-R, -T)$. *Let $\Omega_0$ be the cylindrical domain $\{|x'| < \varepsilon\} \times Tr$. Assume that $a_* R < T$, where $a_* = \sup a_0$ over $\Omega_0$. Let $\Gamma$ be $\partial\Omega_0 \cap \{x_n = -R\}$.*

*Then any solution $u \in H_{(2)}(\Omega_0)$ to the hyperbolic equation $Au = 0$ in $\Omega_0$ with zero Cauchy data on $\Gamma$ is zero on $\Omega_0$.*

PROOF. We will derive this result from Corollary 3.4.6

Let $\delta > 0$. Let us consider the triangle $Tr_\delta$ with vertices at $(-\delta, 0)$, $(-R, -T)$, $(-R, T)$. This triangle is contained in $Tr$, and it is close to it when $\delta$ is small. Let $\Omega_\delta$ be $\{|x'| < \varepsilon_1\} \times Tr_\delta$, where $\varepsilon_1$ depends only on $\varepsilon$ and $\delta$ and is defined below. We will show that $u = 0$ on any $\Omega_\delta$.

Let us assume the opposite. We define the number $\delta^*$ as $-\sup\{x_n^* : u = 0$ on $\Omega_\delta \cap \{x_n < x_n^*\}\}$. Then $\delta_* > \delta$. To obtain a contradiction we will make use of the uniqueness of the continuation for the domain $Con$ defined as the intersection of the two cones $\{|x| < |a_*^{-1}t - \varepsilon|, t < \varepsilon a_*\}$ and $\{|x| < |a_*^{-1}t + \varepsilon|, t > -\varepsilon a_*\}$. We claim that if $Au = 0$ in $Con$ and $u = 0$ on $Con \cap \{|x| < \varepsilon_1\}$ for some $\varepsilon_1$, then $u = 0$ on $Con$.

To prove this we consider any compact set $K$ in $Con$. We find a smooth function $k(t)$ such that $|k'| < a_*^{-1}$, $k$ is positive on the interval $I = (-\varepsilon a_*, \varepsilon a_*)$ and is zero at its endpoints, and $K$ is contained in the set $\{|x| < k(t)\}$. We introduce the family of domains $\Omega_\theta$ defined as $\{|x| < \theta k(t), t \in I\}$. Their boundaries are time-like due to the definition and to the condition $|k'| < a_*^{-1}$. In addition, $\Omega_\theta \subset \{|x| < \varepsilon\}$ when $\theta$ is small and positive, so then $u = 0$ on $\Omega_\theta$. From Corollary 3.4.6 it follows that if $u = 0$ on $\Omega_1$, then in particular, $u = 0$ on $K$, and we have our claim.

Now we go to the basic step of the proof. By extending $u$ as zero on $\{x_n \leq -R\}$ we preserve the differential equation, because the Cauchy data are zero on $\Gamma$. Let us consider the translation of the set $Con$ by $\delta_* + \varepsilon/2$ in the negative direction of the $x_n$-axis. Denote by $Con_*$ any translation of the new set in the $t$-direction such that its upper and lower vertices do not intersect $\partial\Omega_\delta$. We have $Au = 0$ on $Con_*$, and moreover, due to our choice of this cone, $u = 0$ near the $t$-axis of $Con_*$ are less steep than those of $\Omega_\delta$, we conclude that $u = 0$ on $\Omega_\delta \cap \{|x'| < \varepsilon_1, x_n < -\delta_* + \varepsilon_1\}$ for some $\varepsilon_1$ that depends only on $\varepsilon$ and $\delta$. We have a contradiction with the choice of $\delta_*$, which shows that $u = 0$ on $\Omega_\delta$. In particular, $u = 0$ on $\{x' = 0\} \times Tr_\delta$ for any $\delta > 0$, so $u = 0$ on $\{x' = 0\} \times Tr$.

By using translations by $a'$, $|a'| < \varepsilon$, and applying the same argument to the domain $\{|x' - a'| < \varepsilon - |a'|\} \times Tr$, we conclude that $u = 0$ on $\{x' = a'\} \times Tr$. Since the union of these sets over $a'$ is $\{|x'| < \varepsilon\} \times Tr$, the proof is complete. □

It is quite interesting that when the Cauchy data are prescribed on a "large" part $\Gamma$ of the lateral boundary while on the remaining part we have one classical boundary condition, then one can show that the operator mapping the initial data into the lateral Cauchy data is isometric with respect to standard energy norms. So under reasonable conditions, the lateral Cauchy problem is as stable as any classical problem of mathematical physics. However, it seems impossible to obtain an existence theorem (to describe the set of Cauchy data) because in decreasing $\Gamma$ slightly we still will have uniqueness. We start with the case of $\Gamma = \partial G \times (-T, T)$.

We consider a solution $u$ to the boundary value problem

(3.4.7) $Au = f$ in $\Omega = G \times (-T, T)$, $u = 0$ on $\partial G \times (-T, T)$, $\partial G \in C^2$.

We define the energy integral for the hyperbolic equation (3.2.3) as

$$E(t) = 1/2 \int_G ((\partial_t u)^2 + |\nabla u|^2 + u^2)(\,\cdot\,, t).$$

We have the standard energy integral, provided that $u = 0$ on $\partial G$ (Theorem 8.1). This can be proven by multiplying the equation $Au = 0$ by $e^{\tau t}\partial_t u$, integrating over $G \times (0, t)$, and using elementary integral inequalities.

**Theorem 3.4.8.** *Let $\gamma = \partial\Omega$. Let $A$ be a $t$-hyperbolic partial differential operator of second order. Let $\psi$ be pseudo-convex with respect to $A$,*

(3.4.8) $$\psi < 0 \textit{ on } \overline{G} \times \{-T, T\}, \textit{ and } 0 < \psi \textit{ on } \overline{G} \times \{0\}.$$

*Then there is a constant $C$ such that for any solution $u$ to (3.4.7)*

(3.4.9) $$E(t) \leq C\left(\int_{\gamma\times(-T,T)} (\partial_\nu u)^2 + \int_{G\times(-T,T)} f^2\right)$$

*when $-T < t < T$.*

To prove the theorem we will use Carleman estimate (3.2.2′) and the argument of Klibanov and Malinskii [KlM] and of Tataru [Tat1]. Another approach is called the multiplier method which is motivated by the work of Morawetz on mixed-type equations and Friedrich's *abc*-method. We will demonstrate it later.

PROOF. By using Theorem 8.1 and subtracting from $u$ the solution of the boundary value problem (3.4.7) augmented by th zero initial conditions we can assume that

$f = 0$. By the Carleman estimate (3.2.2′)

$$\int_\Omega e^{2\tau\varphi}(|u|^2 + |\nabla u|^2) \leq C(\int_{\partial G\times(-T,T)} e^{2\tau\varphi}|\partial_\nu u|^2 + \int_{G\times\{-T,T\}} e^{2\tau\varphi}(\tau^3|u|^2 + \tau|\nabla u|^2))$$

Since $\varphi = e^{\sigma\psi}$ from the condition (3.4.8) it follows that

$$1 + \varepsilon_1 < \varphi, \text{ on } G \times (-T_1, T_1) \text{ and that } \varphi \leq 1 \text{ on } G \times \{-T, T\}$$

for some small positive $\varepsilon_1, T_1$. Shrinking the integration domain in the left side of the previous inequality, replacing $\varphi$ in the left side by $1 + \varepsilon_1$, by its *supremum* $\Phi$ over the lateral boundary and by 1 in the integrals remaining part of the boundary, and dividing the both sides by $e^{2\tau(1+\varepsilon_1)}$ we obtain

$$\int_{G\times(-T_1,T_1)} (|u|^2 + |\nabla u|^2) \leq C(e^{2\tau\Phi}\int_{\partial G\times(-T,T)} |\partial_\nu u|^2 + e^{-0.5\tau\varepsilon_1} \times \int_{G\times\{-T,T\}} (|u|^2 + |\nabla u|^2)). \tag{3.4.10}$$

where we also used that $\tau^3 e^{-0.5\tau\varepsilon_1} < C$.

The conservation of energy for the hyperbolic problem (Theorem 8.1) implies that

$$C^{-1}E(0) \leq E(t) \leq CE(t) \text{ when } t \in (-T, T).$$

Using these inequalities and the obvious fact that

$$\int_{G\times(-T,T)} (|\partial_t^2 u|^2 + |\nabla u|^2 + |u|^2) = \int_{(-T,T)} E(t)dt$$

we derive from (3.4.10) the bound

$$2T_1C^{-1}E(0) \leq C(e^{2\tau\Phi}\int_{\partial G\times(-T,T)} |\partial_\nu u|^2 + 2Ce^{-0.5\varepsilon_1}E(0)).$$

Choosing $\tau$ so large that $2T_1/C > 2Ce^{-0.5\tau\varepsilon_1}$ we eliminate $E(0)$ from the right side and complete the proof. □

In Carleman estimates (3.2.3′) of Theorem 3.2.1′ one does not need to include all boundary terms. In the papers of Imanuvilov [Im], Tataru [Tat3] the following form of (3.2.3′) was obtained.

**Theorem 3.4.9.** *Let $A$ be a $t$-hyperbolic operator of second order in $\Omega = G \times (-T, T)$. Let a function $\psi$ be pseudo-convex with respect to $A$ on $\overline{\Omega}$ and*

$$\partial_\nu \psi < 0 \text{ on } \Gamma_0. \tag{3.4.11}$$

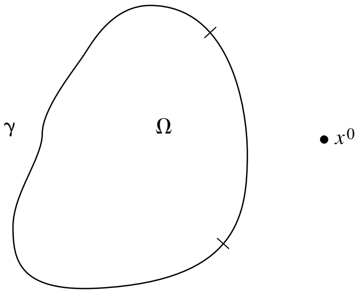


FIGURE 3.4.

*Then there are constants $C_1(\sigma)$, $C_2$ such that*

$$\tau^{3-2|\alpha|}\int_\Omega |\partial^\alpha u|^2 w^2 \leq C_1\Big(\int_\Omega |Au|^2 w^2 + \int_{\partial\Omega\setminus\Gamma_0} \tau|\partial_\nu u|^2 w^2\Big)$$

*when $C_2 < \sigma$, $C_1 < \tau$, for all functions $u \in H_{(2)}(\Omega)$ for which $u = 0$ on $\partial\Omega$, $u = \partial_t u = 0$ on $G \times \{-T, T\}$.*

Using Theorem 3.4.10 instead of Theorem 3.2.1' in the proof of Theorem 3.4.8 one can obtain the bound (3.4.9) with $\gamma = \partial G \setminus \gamma_0$. Now we will obtain a similar result by a different method which can be used for some hyperbolic systems.

Let $\beta$ be a point in $\mathbb{R}^n$. Let $l(x) = x - \beta$. Let $\gamma$ be $\partial G \cap \{l \cdot \nu > 0\}$. The geometry of the problem is illustrated by Figure 3.4

**Exercise 3.4.10.** Let

$$E_a(t) = 1/2\int_G (a_0^2(\partial_t u)^2(,t) + |\nabla u|^2(,t)).$$

Prove that when $b_j = 0$, we have

$$E_a(0) \leq E_a(t) + \varepsilon\int_{G\times(0,t)} (\partial_t u)^2 + C(\varepsilon)\int_{G\times(0,t)} (u^2 + f^2)$$

for any solution $u$ to $a_0^2\partial_t^2 u - \Delta u = f$ satisfying the zero Dirichlet lateral boundary conditions $u = 0$ on $\partial G \times (0, t)$. Here $\varepsilon$ is any positive number. Show that the inequality is valid with $E(0)$ and $E(t)$ interchanged, and moreover when $c = 0$ and $a_0$ does not depend on $t$ one can let $\varepsilon = C(\varepsilon) = 0$.

To solve this exercise we recommend to multiply the equation $Au = f$ by $e^{\tau t}\partial_t u$, integrate by parts over $G \times (0, t)$, and use the elementary inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$, as in the proof of next Theorem.

The following result in a particular case by the multiplier method was first obtained by Lop Fat Ho [Lo].

**Theorem 3.4.11.** *Let $A$ be a hyperbolic partial differential operator of second order. Let $L$ be* $\sup |l|$ *over $G$. Let us assume that*

$$a_0 L < T \text{ and } 0 \leq l \cdot \nabla a_0 \text{ on } G. \tag{3.4.12}$$

*Then there is a constant $C$ such that for any solution $u$ to (3.4.7)*

$$E(t) \leq C\Big(\int_{\gamma\times(-T,T)} (\partial_\nu u)^2 + \int_{G\times(-T,T)} f^2\Big) \tag{3.4.13}$$

*when $-T < t < T$.*

We will give a proof extending the original argument of Lop Fat Ho [Lo] for the wave equation with constant coefficients.

PROOF. We multiply the equation $Au = f$ by the nonstandard factor $l \cdot \nabla u$, integrate by parts using the zero lateral boundary data, and make simple transformations:

$$\begin{aligned}
0 &= \int_G l_k \partial_k u (a_0^2 \partial_t^2 u - \Delta u + cu - f) \\
&= \partial_t \int_G l_k \partial_k u a_0^2 \partial_t u - 1/2 \int_G l_k a_0^2 \partial_k (\partial_t u)^2 \\
&\quad - \int_G l_k \partial_k u \partial_j \partial_j u + \int_G l_k (cu - f) \partial_k u \\
&= \cdots + 1/2 \int_G a_0^2 \partial_k l_k (\partial_t u)^2 + \int_G a_0 (\partial_t u)^2 l_k \partial_k a_0 - \int_{\partial G} l_k \nu_j \partial_k u \partial_j u \\
&\quad + 1/2 \int_G l_k \partial_k (\partial_j u)^2 + \int_G |\nabla u|^2 + \cdots \geq \partial_t \int_G l \cdot \nabla u a_0^2 \partial_t u \\
&\quad + n/2 \int_G a_0^2 (\partial_t u)^2 - 1/2 \int_{\partial G} l_k \nu_k \nu_j \nu_j (\partial_\nu u)^2 \\
&\quad - ((n-2)/2) \int_G |\nabla u|^2 + \int_G (cu - f) l \cdot \nabla u,
\end{aligned}$$

where we denote by $\ldots$ the unchanged terms and use that $\partial_k u = \nu_k \partial_\nu u$ on $\partial G$ due to the lateral boundary condition $u = 0$. We adopt the summation convention over the repeated indices $j, k$ from 1 to $n$. In the last inequality we have used the second condition (3.4.12) and dropped the corresponding nonpositive term. Now, from the definition of $\gamma$ we get

$$\begin{aligned}
&n/2 \int_G a_0^2 (\partial_t u)^2 - ((n-2)/2) \int_G |\nabla u|^2 \leq -\partial_t \int_G a_0^2 l \cdot \nabla u \partial_t u \\
&\quad + 1/2 \int_\gamma l \cdot \nu (\partial_\nu u)^2 + \varepsilon \int_G |\nabla u|^2 + C(\varepsilon) \int_G u^2 + C(\varepsilon) \int_G f^2,
\end{aligned} \tag{3.4.14}$$

where the last two terms bound the integral of $cul \cdot \nabla u$ via the Schwarz inequality and the elementary fact that $2wv \leq \varepsilon w^2 + (1/\varepsilon) v^2$.

We also need a more traditional relation. To obtain it we multiply the hyperbolic equation by $u$ and integrate by parts as above:

$$0 = \int_G u(a_0^2 \partial_t^2 u - \Delta u + cu - f)$$
$$= \partial_t \int_G a_0^2 u \partial_t u - \int_G a_0^2 (\partial_t u)^2 + \int_G |\nabla u|^2 + \int_G cu^2 - \int_G uf.$$

Multiplying this relation by $(n-1)/2$ and subtracting from inequality (3.4.14) we obtain

$$E(t) \le -\partial_t \int_G a_0^2 (l_k \partial_k u + ((n-1)/2)u)\partial_t u + 1/2 \int_\gamma l \cdot \nu (\partial_\nu u)^2$$
$$+ \varepsilon \int_G |\nabla u|^2 + C(\varepsilon) \int_G (u^2 + f^2).$$

We bound $E(t)$ from below using the result of Exercise 3.4.10 and integrate with respect to the time variable over $(-T, T)$ to obtain

$$2TE(0) \le \int_G a_0^2 l \cdot (\partial_t u \nabla u(-T) - \partial_t u \nabla u(T)) + \varepsilon \int_{G\times\{-T,T\}} (\partial_t u)^2$$
$$+ C(\varepsilon) \int_{G\times\{-T,T\}} u^2 + 1/2 \int_{\gamma\times(-T,T)} l \cdot \nu (\partial_\nu u)^2$$
$$+ \varepsilon \int_{G\times(-T,T)} (|\nabla u|^2 + (\partial_t u)^2) + C(\varepsilon) \int_{G\times(-T,T)} (u^2 + f^2).$$

By using the inequality

$$a_0^2 l \cdot \partial_t u \nabla u \le a_0 (L/2)(a_0^2 (\partial_t u)^2 + |\nabla u|^2),$$

letting $a^*$ be $\sup a_0$ over $G$, and bounding $E(t)$ by $E(0)$ as above we obtain

$$2TE(0) \le (2a^* L + 2\varepsilon)E(0) + 1/2 \int_{\gamma\times(-T,T)} l \cdot \nu (\partial_\nu u)^2$$
$$+ \varepsilon C E(0) + C(\varepsilon) \left( \int_{G\times\{-T,T\}} u^2 + \int_{G\times(-T,T)} (u^2 + f^2) \right).$$

By using the simplest trace inequality,

$$u^2(\,, T) \le \varepsilon_1 \int_{(0,T)} (\partial_t u)^2 + C(\varepsilon_1) \int_{(0,T)} u^2,$$

we eliminate the integral over $G \times \{-T, T\}$ possibly increasing $\varepsilon$. Finally, using the first condition (3.4.6), choosing $\varepsilon$, $\varepsilon_1$ so small that $T - a_0 L - (2 + C)\varepsilon - 2\varepsilon_1 C(\varepsilon) > 0$, and putting all terms with $E(0)$ into the left side of the previous inequality for $E(0)$ we arrive at the inequality

$$E(0) \le C \int_{\gamma\times(-T,T)} (\partial_\nu u)^2 + C \int_{G\times(-T,T)} (u^2 + f^2). \tag{3.4.15}$$

The estimate (3.4.13) will follow if we eliminate the last term. We will do it by a standard compactness-uniqueness argument.

Let us assume that the estimate (3.4.12) is not true. Then there is sequence of functions $u_k \in H_{(2)}(\Omega)$, $Au_k = f_k$ such that

$$E(0; u_k) \geq k(\int_{\gamma\times(-T,T)} (\partial_\nu u_k)^2 + \int_{G\times(-T,T)} f_k^2). \tag{3.4.16}$$

Dividing $u_k$ by constants $E(0; u_k)$ we can assume that their energies are equal to 1. From the bound (3.4.14) it follows now that

$$C^{-1} \leq \int_\Omega u_k^2. \tag{3.4.17}$$

Due to the assumption that $E(0; u_k) = 1$ and conservation of the energy, $\|u_k\|_{(1)} \leq C$, so by using compactness of the embedding of $H_{(1)}$ into $L_2$ (Theorem A2) we can extract a subsequence of $u_k$ which is convergent in $L_2(\Omega)$ to $u^*$. We will denote this subsequence by the same symbol $u_k$. Applying (3.4.15) to $u_k - u_m$ and using (3.4.16), (3.4.17) we conclude that $u_k$ is convergent also in $H_{(1)}(\Omega)$. By the definiton (8.0.5) of a weak solution

$$\int_\Omega (-a_0^2 \partial_t u_k \partial\partial_t \phi + \nabla u_k \cdot \nabla\phi) + \int_{\gamma\times(-T,T)} \partial_\nu u_k \varphi = \int_\Omega f_k \phi$$

for any test function $\phi \in C^2(\overline{\Omega})$ which is zero on $G \times \{-T, T\}$ and on $(\partial G \setminus \gamma) \times (-T, T)$. We pass to the limit as $k$ goes to infinity and we obtain the same identity with integral over $\gamma \times (-T, T)$ and $f_k$ replaced by zeros.

To conclude that $u^* = 0$ and to get a contradiction we need Lemma 3.4.7 where a solution is more regular. To achieve this regularity we will use convolution $u_\theta^*$ of $u^*$ with respect to $t$ with the standard mollifying kernel supported in $(-\theta, \theta)$. The functions $u_\theta^*$ are well-defined on $\Omega(\theta) = G \times (-T + \theta, T - \theta)$, they satisfy the above integral identity, they are zero on $\partial G \times (-T, T)$, and $\partial_t^2 u_\theta^* \in L_2(\Omega(\theta))$. The integral identity can be viewed as the definition (4.0.3) of a weak solution $u_\theta^*$ to the elliptic equation $-a_0^2\partial_t^2 u_\theta^* - \Delta u_\theta^* = f$ in $\Omega(\theta)$ where $f = -2a_0^2\partial_t^2 u_\theta^* \in L_2(\Omega(\theta))$. Since $u_\theta^* = 0$ on $\partial G \times (-T + \theta, T - \theta)$ by elliptic boundary regularity (Theorem 4.1) $u_\theta^* \in H_{(2)}(\Omega(2\theta))$. This function solves the hyperbolic equation $Au_\theta^* = 0$ in $\Omega(2\theta)$ and it has zero Cauchy data on $\gamma \times (-T + 2\theta, T - 2\theta)$. Using condition (3.4.12) and choosing $\theta$ small we can satisfy conditions of Lemma 3.4.7. By this lemma $u_\theta^* = 0$ is zero on $G \times (-t_0, t_0)$ for small positive $t_0$. Since $u_\theta^* = 0$ on $\partial G \times (-T + 2\theta, T - 2\theta)$ by theory of hyperbolic problems (section 8.1) it is zero on $\Omega(2\theta)$. Letting $\theta$ go to 0 we conclude that $u^* = 0$ on $\Omega$. This is a contradiction, because due to (3.4.17) and to the $L_2(\Omega)$-convergence of $u_k$ to $u^*$ the limit is not zero.

The contradiction shows that the estimate (3.4.13) holds. □

**Exercise 3.4.12.** Assume that $\gamma$ is an open (nonempty) part of $\partial G$. Let $L_\gamma$ be $\sup\inf|l(x)|$(inf over $l$ and sup over $x \in G$), where $l(x)$ is a smooth curve in $G$ joining $x$ and a point of $\gamma$. (In other words, $L_\gamma$ is the "interior" distance from $\gamma$ to $G$. It coincides with the Hausdorff distance when $G$ is convex.)

Prove that if $a_0 L_\gamma < T$, then any solution $u \in H_{(2)}(\Omega)$ to the equation $Au = 0$ in $\Omega$ with zero Cauchy data on $\gamma \times (-T, T)$ is zero on $G \times \{0\}$.

Observability inequality (3.4.9), its implications for stabilization of solutions of hyperbolic initial boundary value problems and to exact controllabity also in more difficult case of zero Neumann boundary data on $\partial G$ as well as many references to control theory and the multiplier method are given in papers of Imanuvilov [Im], Lasiecka, Triggiani, Yao and Zhang [LaTZ1], [LaTY], [TrY].

By using methods of geometrical optics and Fourier analysis and assuming some nondegeneracy condition about $\partial G$, Bardos, Lebeau, and Rauch [BarLR] obtained necessary and sufficient conditions for the set $\Gamma$ when Lipschitz stability in the Cauchy problem holds.

Now we prove and formulate similar results for the Schrödinger-type equation, which possesses properties of the both parabolic and hyperbolic equations.

In Theorem 3.4.13 we consider the operator $Au = ia_0\partial_t u - \partial_1^2 u - \cdots - \partial_n^2 u + \sum b_j \partial_j u + cu$.

**Theorem 3.4.13.** *Let us consider a domain G and a surface $\gamma$ satisfying conditions* (3.4.3). *Suppose that*

(3.4.18) $$-2a_0 < x \cdot \nabla a_0, \quad \partial_n a_0 \leq 0 \text{ on } \Omega, \Omega = G \times (-T, T).$$

*Then for any domain $\Omega_\varepsilon$ with closure in $\Omega \cup \Gamma$ we have uniqueness of a solution u to the Cauchy problem* (3.2.5) *and the estimate*

$$\|\partial_j u\|_2(\Omega_\varepsilon) \leq C(F + M^{1-\kappa} F^\kappa),\ j = 1, \ldots, n,$$

*where $C$, $\kappa \in (0, 1)$ depend on $\Omega_\varepsilon$, $M$ is $\| \sum \partial_k u\|_2(\Omega) + \|u\|_2(\Omega)$, and $F$ is as defined in Theorem* 3.2.2.

PROOF. Due to the assumptions, $\Omega_\varepsilon \subset \{|t| < T - \tau\}$ for some positive $\tau$. Let us choose a $C^\infty$-function $\omega(t)$ that is 0 when $|t| < T - \tau$, $0 \leq \omega \leq h$ everywhere, and $\omega(T) = \omega(-T) = h$. The substitution

$$x_1 = y_1, \ldots, x_{n-1} = y_{n-1}, x_n = y_n - \omega(t), t = s$$

does not change the principal part of the operator $A$ and the second condition (3.4.18), while the first one is replaced by $-2a_0 < y \cdot \nabla a_0 - \omega \partial_n a_0$. In the new variables, $\Gamma$ contains $\partial\Omega \cap \{y_n < 0\}$ and $\Omega_\varepsilon$ is a subset of $\{|y'| < r, -h < y_n < 0\}$. Later on we will switch to the notation $x$ again, keeping in mind the change in condition (3.4.18).

We will use again the weight function $\varphi = \exp(\sigma/2\psi)$ with

$$\psi(x, t) = x_1^2 + \cdots + x_{n-1}^2 + (x_n - \beta_n)^2 - \beta_n^2 - r^2, h^2 + 2h\beta_n > r^2$$

and check for this function the strong pseudo-convexity condition (3.2.2) with $m = (2, \ldots, 2, 1)$ and $q = n$.

The equality $A_m(\zeta) = 0$ for $\zeta = \xi + i\tau\nabla_n\varphi$, $\nabla_n\varphi = \sigma/2\varphi\nabla\psi$ is equivalent to the relations for its real and imaginary parts:

(3.4.19) $$-a_0\xi_{n+1} + \sum \xi_j^2 = \tau^2\sigma^2 4^{-1}\varphi^2|\nabla\psi|^2,\ 0 = \xi \cdot \nabla\psi.$$

The left side of (3.2.2) is the sum of

$$\mathfrak{H}_1 = 4\sigma\varphi(|\zeta_1|^2 + \cdots + |\zeta_n|^2) + \tau^{-1}\Im(\partial_1 a_0 \xi_{n+1} 2\zeta_1 + \cdots + \partial_n a_0 \xi_{n+1} 2\zeta_n)$$

and of

$$\mathfrak{H}_2 = \sigma^2\varphi|\partial_1\psi\zeta_1 + \cdots + \partial_n\psi\zeta_n|^2.$$

By using the formula for $\zeta$ and the relations (3.4.19), one can see that

$$\begin{aligned}\mathfrak{H}_1 &= \sigma\varphi(4|\xi|^2 + \tau^2\sigma^2\phi^2|\nabla\psi|^2 - 2\xi_{n+1}x\cdot\nabla a_0 + 2\xi_{n+1}\beta_n\partial_n a_0)\\ &= \sigma\phi(2a_0^{-1}(2a_0 + x\cdot\nabla a_0 - \beta_n\partial_n a_0)|\xi|^2\\ &\quad + a_0^{-1}(2a_0 - x\cdot\nabla a_0 + \beta_n\partial_n a_0)2^{-1}\tau^2\sigma^2\varphi^2|\nabla\psi|^2),\\ \mathfrak{H}_2 &= \sigma^4\varphi^2\tau^2|\nabla\psi|^2.\end{aligned}$$

Using homogeneity and assuming $|\zeta| = 1$ and considering $\tau = 0$ we conclude that condition (3.4.18) and the first relation (3.4.19) guarantee that $\mathfrak{H}_1 > 0$ when $\beta_n$ is large enough, and then $\mathfrak{H} > 0$. At this point we fix $\beta_n$. This inequality is preserved by continuity when $|\tau| < \varepsilon_0$. When $|\tau| \geq \varepsilon_0$ we can achieve positivity again by choosing $\sigma$ large. So $\varphi$ is strongly pseudo-convex, and one can apply Theorem 3.2.2. To do so we define $\Omega_\delta^*$ as $\Omega \cap \{\psi > \delta\}$. By choosing $\beta_n$ large we can achieve that $\Omega_\varepsilon$ is contained in $\Omega_\delta^*$ for some small $\delta$. Our choice of $\psi$ and the conditions on $\beta_n$ guarantee that $\psi < 0$ on $\partial\Omega\backslash\Gamma$. Now the claim follows from Theorem 3.2.2. □

Observe that positive $T$ can be arbitrarily small. Conditions (3.4.18) are satisfied when $a_0$ is constant. When $\gamma = \partial G$, we can use a translation in the $x_n$-direction and achieve that $G \subset \{x_n < 0\}$, so the conditions of this theorem will be satisfied, and we have uniqueness and stability of the continuation from the whole lateral boundary $\partial G \times (-T, T)$ for any (small) $T$. Carleman estimates with semi-explicit boundary terms and their applications to observability with Dirichlet or Neumann boundary conditions are obtained in the recent paper by Lasiecka, Triggiani, and Zhang [LaTZ2].

A general scheme of deriving controllabity/observability from Carleman estimates, conservation of energy, and propagation of singularities is given by Littman and Taylor [LitT].

## 3.5 Systems of partial differential equations

The results of sections 3.2-3.4 are valid for the Cauchy problem for principally diagonal systems

(3.5.1) $\quad \mathbf{Au} = \mathbf{f} \text{ in } \Omega,\ (\mathbf{A} = \mathbf{A}(d) + \mathbf{A}_1)\ \mathbf{u} = \mathbf{g}_0,\ \partial_\nu\mathbf{u} = \mathbf{g}_1 \text{ on } \Gamma$

where $\mathbf{A}(d)$ is a diagonal second-order matrix partial differential operator with $C^1(\overline{\Omega})$-coefficients and $\mathbf{A}_1$ is a matrix first-order operator with $L_\infty(\Omega)$-coefficients. First we observe that Theorem 3.2.1$'$ implies

**Corollary 3.5.1.** *Let a function $\psi$ be pseudo-convex with respect to all scalar operators forming* $\mathbf{A}(d)$.

*Then there are constants $C_1(\sigma)$ and $C_2$ such that for $\sigma > C_2$, $\tau > C_1(\sigma)$ and all vector-functions* $\mathbf{u} \in H_{(1)}(\Omega)$

$$\int_\Omega e^{2\tau\varphi}(\tau^3|\mathbf{u}|^2 + \tau|\mathbf{u}|^2) \leq C_2(\int_\Omega e^{2\tau\varphi}|\mathbf{A}\mathbf{u}|^2 + \int_{\partial\Omega} e^{2\tau\varphi}(\tau^3|\mathbf{u}|^2 + \tau|\mathbf{u}|^2)).$$

Using Corollary 3.5.1 in the proof of Theorem 3.2.2 instead of Theorem 3.2.1 one obtains

**Theorem 3.5.2.** *Let a function $\psi$ be pseudo-convex with respect to all scalar operators forming the principal part of* $\mathbf{A}$. *Let $\psi < 0$ on* $\partial\Omega \setminus \Gamma$.

*Then there are constants $C, \kappa$ depending on $\Omega, \Gamma, \mathbf{A}, \varphi, \varepsilon$ such that a solution* $\mathbf{u}$ *to the Cauchy problem* (3.5.1) *satisfies the bound*

$$\|\mathbf{u}\|_{(1)}(\Omega_\varepsilon) \leq C(F + \|\mathbf{u}\|_{(1)}^{1-\kappa}(\Omega_0)F^\kappa) \tag{3.5.2}$$

*where* $F = \|\mathbf{f}\|_{(0)}(\Omega_0) + \|\mathbf{u}\|_{(0)}(\Gamma) + \|\nabla\mathbf{u}\|_{(0)}(\Gamma))$.

This result gives better stability estimate compared with Theorem 5.3.2 gaining 0.5 in indices of boundary norms.

**Theorem 3.5.3.** *Let* $\mathbf{A}$ *be a principally diagonal $t$-hyperbolic system with time independent coefficients. Let the surface $\Gamma$ be non-characteristic with respect to all scalar operators forming the principal part of* $\mathbf{A}$.

*Then there is an open subset* $\Omega(0)$ *of* $\Omega$ *whose closure contains* $\Gamma$ *such that a solution to the Cauchy problem* (3.5.1) *is unique in* $\Omega(0)$.

Complete proofs of Theorems 3.5.2, 3.5.3 are given in the paper of Eller, Isakov, Nakamura, and Tataru [EINT]. It was also observed in this paper that Theorems 3.5.2, 3.5.3 imply uniqueness results in the lateral Cauchy problem for the classical isotropic Maxwell's and elasticity systems.

First we consider the dynamical Maxwell's system

$$\partial_t(\varepsilon\mathbf{E}) = curl\ \mathbf{H} + \mathbf{j},$$
$$\partial_t(\mu\mathbf{H}) = curl\ \mathbf{E},$$
$$div(\varepsilon\mathbf{E}) = 4\pi\rho,\ div(\mu\mathbf{H}) = 0 \tag{3.5.3}$$

for the electric and magnetic fields $\mathbf{E} = (E_1, E_2, E_3), \mathbf{H} = (H_1, H_2, H_3)$ in the medium with electric permittivity and magnetic permeability $\varepsilon, \mu \in C^2(\overline{\Omega})$, the density of electrical current $\mathbf{j}(x,t)$, and the electrical charge density $\rho(x,t)$. Here $\Omega$ is a bounded domain in $\mathbb{R}^4$.

Differentiating the first equation (3.5.3) with respect to $t$, applying the curl to the second equation and using it to replace $\partial_t curl\mathbf{H}$ in the first equation we obtain

$$\partial_t^2(\varepsilon\mathbf{E}) + 1/\mu(curlcurl\mathbf{E} + \partial_t\mu curl\mathbf{H} + \partial_t(\nabla\mu \times \mathbf{H})) = \partial_t\mathbf{j}.$$

Similarly,

$$\partial_t^2(\mu\mathbf{H}) + 1/\varepsilon(curl curl\mathbf{H} - \partial_t\varepsilon curl\mathbf{E} - \partial_t(\nabla\varepsilon \times \mathbf{E}) + curl\mathbf{j}) = 0.$$

Using that $curl\, curl = -\Delta + \nabla div$ and utilizing the last two equations in (3.5.3) to substitute

$$div\mathbf{E} = 1/\varepsilon(4\pi\rho - \nabla\varepsilon \cdot \mathbf{E}),\ div\mathbf{H} = -1/\mu(\nabla\mu \cdot \mathbf{H}),$$

we conclude that the Maxwell's system (3.5.3) implies the following $6 \times 6$ principally diagonal system

$$\varepsilon\mu\partial_t^2\mathbf{E} - \Delta\mathbf{E} + \mathbf{A}_{E,1} = \mu\partial_t\mathbf{j} - \nabla((4\pi\rho)/\varepsilon),$$

(3.5.4) $$\varepsilon\mu\partial_t^2\mathbf{H} - \Delta\mathbf{H} + \mathbf{A}_{H,1} = -curl\mathbf{j},$$

where we introduced the matrix operators of first order

$$\mathbf{A}_{E,1}(\mathbf{E},\mathbf{H}) = 2\mu\partial_t\varepsilon\partial_t\mathbf{E} + \mu\partial_t^2\varepsilon\mathbf{E} + \partial_t(\nabla\mu \times \mathbf{H}) + \partial_t\mu curl\mathbf{H} - \nabla(1/\varepsilon\nabla\varepsilon \cdot \mathbf{E}),$$

$$\mathbf{A}_{H,1}(\mathbf{E},\mathbf{H}) = 2\varepsilon\partial_t\mu\partial_t\mathbf{H} + \varepsilon\partial_t^2\mu\mathbf{H} - \nabla(\mu\nabla\mu \cdot \mathbf{H}) - \partial_t\varepsilon curl\mathbf{E} - \partial_t(\nabla\varepsilon \times \mathbf{E}).$$

Specifying Theorems 3.5.2, 3.5.3 to the system (3.5.4) we obtain two uniqueness of the continuation results for the Cauchy problem for the Maxwell's system (3.5.3).

**Corollary 3.5.4.** *Let a function $\psi$ be pseudo-convex with respect to the wave operator $\varepsilon\mu\partial_t^2 - \Delta$ in $\overline{\Omega}$. Let $\psi < 0$ on $\partial\Omega \setminus \Gamma$ where $\Gamma$ is a $C^2$- (hyper)surface in $\mathbb{R}^4$.*

*Then there are constants $C, \kappa \in (0,1)$ depending on $\Gamma, \varepsilon, \mu, \psi, \delta$ such that for any solution $(\mathbf{E},\mathbf{H})$ to the Maxwell's system (3.5.3) we have*

(3.5.5) $$\|\mathbf{E}\|_{(1)}(\Omega_\delta) + \|\mathbf{H}\|_{(1)}(\Omega_\delta) \le C(F + M^{1-\kappa}F^\kappa)$$

*where* $F = \|\mathbf{E}\|_{(1)}(\Gamma) + \|\mathbf{H}\|_{(1)}(\Gamma) + \|\mathbf{j}\|_{(1)}(\Omega) + \|\rho\|_{(1)}(\Omega)$ *and* $M = \|\mathbf{E}\|_{(1)}(\Omega) + \|\mathbf{H}\|_{(1)}(\Omega)$.

Observe that to get the Cauchy data on $\Gamma$, one has to use $\mathbf{E}$, $\mathbf{H}$ and the equations (3.5.3) to obtain normal components of $\nabla\mathbf{E}$, $\nabla\mathbf{H}$ on $\Gamma$.

**Corollary 3.5.5.** *Let the coefficients $\varepsilon, \mu$ be time independent. Let a surface $\Gamma$ be non-characteristic with respect to the wave operator $\varepsilon\mu\partial_t^2 - \Delta$, $\mathbf{j} = \mathbf{0}$, $\rho = 0$ in $\Omega$, and $\mathbf{E} = \mathbf{H} = \mathbf{0}$ on $\Gamma$.*

*Then there is a neighborhood $V$ of $\Gamma$ such that $\mathbf{E} = \mathbf{H} = \mathbf{0}$ in $\Omega \cap V$.*

**Exercise 3.5.6.** By using first two equations (3.5.3) to substitute $curl\mathbf{H}$, $\partial_t\mathbf{H}$ onto the first equation (3.5.4) show that any solution to (3.5.3) satisfies

$$\varepsilon\mu\partial_t^2\mathbf{E} - \Delta\mathbf{E} + 2\mu\partial_t\varepsilon\partial_t\mathbf{E} + \mu\partial_t^2\varepsilon\mathbf{E} - \nabla(1/\varepsilon\nabla\varepsilon \cdot \mathbf{E}) + \partial_t\mu\partial_t(\varepsilon\mathbf{E})$$
$$+ 1/\mu\nabla\mu \times curl\mathbf{E} + 1/(2\mu)\partial_t(\nabla\mu^2) \times \mathbf{H})$$
$$= \partial_t(\mu\mathbf{j}) - \nabla((4\pi\rho)/\varepsilon)$$

In particular, when $\mu$ does not depend on time the equations for $\mathbf{E}$ do not involve $\mathbf{H}$, i.e. the system is uncoupled.

A similar equation holds for $\mathbf{H}$.

Now we will consider the dynamical isotropic elasticity system

$$\rho\partial_t^2\mathbf{u} - \mu\Delta\mathbf{u} - \mu\nabla div\mathbf{u} - \nabla(\lambda div\mathbf{u}) - \sum_{j=1}^{3}\nabla\mu\cdot(\nabla u_j + \partial_j\mathbf{u})\mathbf{e}_j = \mathbf{F} \tag{3.5.6}$$

for the displacement vector $\mathbf{u} = (u_1, u_3, u_3)$ of an elastic medium $\Omega$ of density $\rho \in C^1(\overline{\Omega})$ with the Lame parameters $\lambda, \mu \in C^2(\overline{\Omega})$. For results on initial boundary value problems for the system (3.5.6) we refer to Ciarlet [Ci].

To principally diagonalize the system (3.5.6) so we introduce the functions

$$div\mathbf{u} = v,\ curl\mathbf{u} = \mathbf{w}. \tag{3.5.7}$$

Dividing the equations (3.5.5) by $\rho$ and applying the operator $div$ to the both parts of the resulting equality we obtain

$$\partial_t^2 v - (\lambda + 2\mu)/\rho\Delta v - \nabla(\mu/\rho)\cdot\Delta\mathbf{u} - \nabla(\mu/\rho)\cdot\nabla v - 2/\rho\nabla\lambda\cdot\nabla v - \Delta\lambda/\rho v -$$
$$\nabla 1/\rho\cdot\nabla(\lambda v) - \sum_{j=1}^{3}\partial_j(\nabla\mu)/\rho\cdot(\nabla u_j + \partial_j\mathbf{u}) - \nabla\mu/\rho\cdot(\nabla v + \Delta\mathbf{u}) = div(\mathbf{F}/\rho)$$

and using the known identity $\Delta = \nabla div - curlcurl$ we obtain the equation

$$\partial_t^2 v - (\lambda + 2\mu)/\rho\Delta v + A_{1;4}\mathbf{U} = div(\mathbf{F}/\rho) \tag{3.5.8}$$

where $\mathbf{U} = (\mathbf{u}, v, \mathbf{w})$ and

$$\begin{aligned} A_{4;1}\mathbf{U} = &-2(\nabla(\mu/\rho) + (\nabla(\lambda+\mu))/\rho)\cdot\nabla v - \nabla(1/\rho)\cdot(\lambda v)\\ &+ (\nabla(\mu/\rho) + \nabla\mu/\rho)\cdot curl\mathbf{w}\\ &- \sum_{j=1}^{3}\partial_j(\nabla\mu/\rho)\cdot(\nabla u_j + \partial_j\mathbf{u}) - \Delta\lambda/\rho v. \end{aligned}$$

Similarly, applying the $curl$ and using that $curl(f\mathbf{u}) = fcurl\mathbf{u} + \nabla f\times\mathbf{u}$ after relatively lenghty but standard computations we obtain

$$\partial_t^2\mathbf{w} - \mu/\rho\Delta\mathbf{w} + \mathbf{A}_{5;1}\mathbf{U} = curl(\mathbf{F}/\rho), \tag{3.5.9}$$

where

$$\mathbf{A}_{5;1} = -2\nabla(\mu/\rho)\times\nabla v + \nabla(\mu/\rho)\times curlw - \nabla(1/\rho)\times\nabla(\lambda v)$$

$$-(\partial_2(\nabla\mu/\rho)\cdot(\nabla u_3 + \partial_3\mathbf{u}) - \partial_3(\nabla\mu/\rho)\cdot(\nabla u_2 + \partial_2\mathbf{u}) - \nabla\mu/\rho\cdot\nabla w_1)\mathbf{e}_1$$

$$-(\partial_3(\nabla\mu/\rho)\cdot(\nabla u_1 + \partial_1\mathbf{u}) - \partial_1(\nabla\mu/\rho)\cdot(\nabla u_3 + \partial_3\mathbf{u}) - \nabla\mu/\rho\cdot\nabla w_2)\mathbf{e}_2$$

$$-(\partial_1(\nabla\mu/\rho)\cdot(\nabla u_2 + \partial_2\mathbf{u}) - \partial_2(\nabla\mu/\rho)\cdot(\nabla u_1 + \partial_1\mathbf{u}) - \nabla\mu/\rho\cdot\nabla w_3)\mathbf{e}_3.$$

So we have for $\mathbf{U} = (u_1, u_2, u_3, v, w_1, w_2, w_3)$ the principally diagonal system (3.5.5), (3.5.8), (3.5.9) where the diagonal entries of the diagonal of the principal part are the isotropic wave operators $\partial_t^2 - \mu/\rho\Delta$ except for the equation (3.5.7) where the principal part is the wave operator $\partial_t^2 - (\lambda + 2\mu)/\rho\Delta$.

The principal diagonalization (3.5.6), (3.5.8), (3.5.9) of the isotropic elasticity system and Theorem 3.5.2 imply

**Corollary 3.5.7.** *Let a function $\psi$ be pseudo-convex with respect to the wave operators $\rho/\mu\partial_t^2 - \Delta$, $\rho/(\lambda + 2\mu)\partial_t^2 - \Delta$ in $\overline{\Omega}$. Let $\psi < 0$ on $\partial\Omega \setminus \Gamma$.*

*Then there are constants $C, \kappa \in (0, 1)$, depending on $\delta$ such that for any solution $\mathbf{u}$ to the elasicity system (3.5.6)*

$$\|\mathbf{u}\|_{(1)}(\Omega_\delta) \le C(F + \|\mathbf{u}\|_{(2)}(\Omega_0)^{1-\kappa} F^\kappa) \tag{3.5.10}$$

*where $F = \|\mathbf{f}\|_{(0)}(\Omega_0) + \|\mathbf{u}\|_{(2)}(\Gamma) + \|\partial_\nu \mathbf{u}\|_{(1)}(\Gamma)$*

Similarly, Theorem 3.5.3 implies

**Corollary 3.5.8.** *Let the coefficients $\rho, \mu, \lambda$ be time independent. Let a surface $\Gamma$ be non-characteristic with respect to the wave operators $\rho/\mu\partial_t^2 - \Delta$, $\rho/(\lambda + 2\mu)\partial_t^2 - \Delta$.*

*Then a solution to the Cauchy problem for the elasticity system with the Cauchy data on $\Gamma$ is unique in $\Omega$ near $\Gamma$.*

This sharp uniqueness of the continuation result was used by McLaughlin and Yoon [McLY] to get first uniqueness result for $\mu, \lambda$ in the so-called elastic sonography, when one recovers elastic parameters from a complete knowledge of the displacement $\mathbf{u}$ inside $Q$.

Recently, Imanuvilov, Isakov and Yamamoto [IIY] obtained a most natural Carleman estimate for the elasticity system on compactly supported functions.

**Theorem 3.5.9.** *Let the function $\psi \in C^3(\overline{\Omega})$ be pseudo-convex with respect to the wave operators $\rho/\mu\partial_t^2 - \Delta$, $\rho/(\lambda + 2\mu)\partial_t^2 - \Delta$ in $\overline{\Omega}$.*

*Then there are constants $C_1(\sigma)$, $C_2$ such that for $C_2 < \sigma$ and $C_1 < \tau$*

$$\int_\Omega e^{2\tau\varphi}(\tau^2|\mathbf{u}|^2 + \tau(|\operatorname{div}\mathbf{u}|^2 + |\operatorname{curl}\mathbf{u}|^2)) \le C\int_\Omega e^{2\tau\varphi}|\mathbf{A}_e\mathbf{u}|^2 \tag{3.5.11}$$

*for all functions $\mathbf{u} \in C_0^2(\Omega)$. Here $\mathbf{A}_e\mathbf{u}$ is the left side in (3.5.6).*

By using the same new unknown functions $v$, $\mathbf{w}$ (3.5.7) and Carleman estimates for the Laplace operator given in section 3.3 one can obtain uniqueness of the continuation results for time independent solutions to the elasticity system (3.5.6) under the same regularity assumptions on its coefficients [AITY].

There are important systems which can not be principally diagonalized but have a special "upper triangular" principal part. As an example we consider the system of thermoelasticity

$$\mathbf{A}_e\mathbf{u} + \mathbf{A}_{1;1}(\mathbf{u}, v) = 0,$$

$$\partial_t v - \sum_{j,k=1}^{3} a_{jk}\partial_j\partial_k v + A_{2;1}(div\mathbf{u}, curl\mathbf{u}, \mathbf{u}, v) = 0 \tag{3.5.12}$$

for the displacement vector $\mathbf{u}$ and the temperature $v$. Here $\mathbf{A}_{1;1}$ is a linear matrix partial differential operator of first order with respect to $v$ an of zero order with respect to $\mathbf{u}$ with $C^1(\overline{\Omega})$-coefficients, the symmetric matrix $(a_{jk}) \in C^1(\overline{\Omega})$ and is positive in $\overline{\Omega}$, and $A_{2;1}$ is a first order linear partial differential operator with $L_\infty(\Omega)$-coefficients. Here $\Omega$ is a bounded domain in $\mathbb{R}^4$. Other examples include von Karman system and systems for elastic plates [Is13], [Is14 ].

**Theorem 3.5.10.** *Let a function $\psi$ be pseudo-convex with respect to the wave operators $\rho/\mu\partial_t^2 - \Delta$, $\rho/(\lambda + 2\mu)\partial_t^2 - \Delta$ in $\overline{\Omega}$. Let $\Gamma \subset \partial\Omega$ be a $C^2$-(hyper)surface and $\psi < 0$ on $\partial\Omega \setminus \Gamma$.*

*Then there are constants $C, \kappa$, depending on $\delta$ such that for any solution $(\mathbf{u}, v)$ to the thermoelasicity system (3.5.12)*

$$\|\mathbf{u}\|_{(1)}(\Omega_\delta) + \|v\|(\Omega_\delta) \leq C(F + \|\mathbf{u}\|_{(2)}(\Omega_0)^{1-\kappa}F^\kappa)$$

*where* $F = \|\mathbf{f}\|_{(0)}(\Omega_0) + \|\mathbf{u}\|_{(2.5)}(\Gamma) + \|\partial_\nu\mathbf{u}\|_{(1.5)}(\Gamma) + \|v\|_{(1.5)}(\Gamma) + \|\nabla v\|_{(0.5)}(\Gamma)$.

A proof is given in [ElI]. We will outline basic ideas of this proof.

Using (3.5.12), from (3.5.6),(3.5.8), (3.5.9) with $\mathbf{F} = -\mathbf{A}_{1;1}(\mathbf{u}, v)$ we have

$$\mathbf{A}(d)\mathbf{U} = \mathbf{A}_1\mathbf{U} + \mathbf{A}_2 v \text{ in } \Omega \tag{3.5.13}$$

where $\mathbf{A}(d)$ is a diagonal $7 \times 7$ matrix linear partial differential operator with diagonal operators $\rho/\mu\partial_t^2 - \Delta$ or $\rho/(\lambda + 2\mu)\partial_t^2 - \Delta$, $\mathbf{U} = (\mathbf{u}, div\mathbf{u}, curl\mathbf{u})$, and $\mathbf{A}_1, \mathbf{A}_2$ are linear partial differential operators of first and second order with $L_\infty(\Omega)$-coefficients.

Let us introduce a cut-off function $\chi \in C^\infty(\mathbf{R}^4)$, $\chi = 1$ on $\Omega_{2\varepsilon}$, $\chi = 0$ on $\Omega \setminus \Omega_\varepsilon$. Defining $\mathbf{U}^0 = \chi\mathbf{U}$, $v^0 = \chi v$ and using the Leibniz' formula we derive from (3.5.12), (3.5.13) that

$$\mathbf{A}(d)\mathbf{U}^0 = \chi\mathbf{A}_2 v^0 + \mathbf{A}_{2;1}(\mathbf{U}, v)$$

$$\partial_t v^0 - \sum A_{jk}\partial_j\partial_k v^0 = A_{3;1}(\mathbf{U}, v) \tag{3.5.14}$$

where $\mathbf{A}_{2;1}, A_{3;1}$ are (matrix) linear partial differential operators with $L_\infty(\Omega)$-coefficients. Moreover, $\mathbf{A}_{2;1}$ is of second order but it does not involve $\partial_t^2 v$ and second order derivatives of $\mathbf{U}$, and $A_{3;1}$ is of first order and does not involve $\partial_t v$.

Using the Carleman estimate of Theorem 3.4.3 for each of the seven first equations of (3.5.14) and summing the results over components we yield

$$\int_\Omega (\sigma\tau\varphi)^{3-2|\alpha|} e^{2\tau\varphi} |\partial^\alpha \mathbf{U}^0|^2$$

$$\leq C \int_\Omega e^{2\tau\varphi} \Big( \sum_{|\alpha|\leq 1} |\partial^\alpha \mathbf{U}|^2 + \sum_{|\beta|\leq 2, \beta_{n+1}\leq 1} |\partial^\beta v^0|^2 + \sum_{|\alpha|\leq 1, \alpha_{n+1}=0} |\partial^\alpha v|^2 \Big).$$

Similarly, from Theorem 3.3.12 (applied to $(\sigma\tau\varphi)^{1/2} v^0$) and from the last equation (3.5.14) we have

$$\sigma \int_\Omega (\sigma\tau\varphi)^{4-2|\alpha|} e^{2\tau\varphi} |\partial^\alpha v^0|^2$$

$$\leq C \int_\Omega (\sigma\tau\varphi) e^{2\tau\varphi} \Big( \sum_{|\alpha|\leq 1} |\partial^\alpha \mathbf{U}|^2 + \sum_{|\beta|\leq 1, \beta_{n+1}=0} |\partial^\beta v|^2 \Big).$$

We can choose $\tau$ large and use the choice of $\chi$ to eliminate the integrals over $\Omega_{2\varepsilon}$ in the right sides. Denoting by $\dots$ terms bounded by $e^{4\tau\varepsilon}$ and bounding the terms with $\mathbf{U}$ in the right side of the previous inequality by preceding inequality we yield

$$\sigma \int_{\Omega_{2\varepsilon}} \sum_{|\alpha|\leq 2, \alpha_{n+1}\leq 1} (\sigma\tau\varphi)^{4-2|\alpha|} e^{2\tau\varphi} |\partial^\alpha v|^2 \leq C \int_{\Omega_{2\varepsilon}} e^{2\tau\varphi} \sum_{|\alpha|\leq 2, \alpha_{n+1}\leq 1} |\partial^\alpha v|^2 + \cdots$$

Dividing the both parts by $e^{4\tau\varepsilon}$ and choosing $\sigma$ and $\tau$ large we conclude that $v = 0$ in $\Omega_{2\varepsilon}$. Similarly, $\mathbf{U} = 0$ in $\Omega_{2\varepsilon}$ for any positive $\varepsilon$. This completes the proof of uniqueness of $(\mathbf{u}, v)$ in $\Omega_0$. Stability estimate can be obtained as in the proof of Theorem 3.2.2.

Uniqueness of the continuation results for the thermoelasticity system are used in control theory. For some generalisations and applications we refer to the paper of Eller, Lasiecka, and Triggiani [ElLT].

## 3.6 Open problems

**Problem 3.1.** Prove backward uniqueness of a (regular) solution to the general parabolic equation of second order in $Q$ with the oblique derivative lateral boundary condition

$$\sum l_j \partial_j u + bu = 0 \text{ on } \partial\Omega \times (0, T),$$

where $l_j, b \in C^2(\partial\Omega \times [0, T])$ and the vector field $l$ is not tangent on $\partial\Omega : l \cdot \nu > \varepsilon > 0$.

If $l = \nu$ (Neumann type condition), then uniqueness follows from Theorem 3.3.1. If the coefficients of $A$ and of the boundary condition do not depend on $t$, uniqueness can be derived by methods of analytic semigroups as described in Section 3.1. In Theorem 3.1.3 one assumes "essential self-adjointness" of the operator $A$, which is wrong in the case of an oblique derivative boundary condition.

There is another aspect of the uniqueness problem related to parabolic equations with changing direction of time as in Example 3.1.8.

**Problem 3.2.** Prove that a smooth ($C^{2,1}(\overline{\Omega})$) solution to the equation

$$\partial_t u - \operatorname{div}(a\nabla u) = 0 \text{ in } Q$$

satisfying zero initial and lateral boundary conditions

$$u = 0 \text{ on } \Omega \times \{0\}, u = 0 \text{ on } \partial\Omega \times (0, T)$$

is zero in $Q$, provided that $\nabla a \neq 0$ when $a = 0$, $a \in C^1(\overline{\Omega})$.

The counterexamples to the uniqueness of the continuation for hyperbolic equations described in Section 3.4 do not cover the simplest perturbation of the wave equation, and the next problem is related to this situation.

**Problem 3.3.** Find real-valued functions $u$ and $c$ in $C^\infty(\mathbb{R}^3)$ such that

$$\partial_t^2 u - \Delta u + cu = 0 \text{ in } \mathbb{R}^3$$

and $\operatorname{supp} u = \{x_2 \leq 0\}$.

When $c$ does not depend on $t$ such a counterexample is impossible due to the recent results of Tataru: According to Corollary 3.4.6, if $u = \partial_2 u = 0$ when $x_2 = 0$, then $u = 0$ near the time-like plane $\{x_2 = 0\}$. On the other hand, there are counterexamples with complex first-order perturbations mentioned in Section 3.4 and even (recent) counterexamples with complex-valued zero-order perturbations $c(x, t)u$ given by Alinhac and Baouendi [AliB]. Complex-valued coefficients generate multiple characteristics that could cause nonuniqueness, and it is not clear how crucial is high multiplicity. The real-valued case looks like a harder one, and we are aware only of the counterexample mentioned in Section 3.4.

**Problem 3.4.** Prove that a $C^2$-smooth solution $u$ to the nonlinear wave equation $\partial_t^2 u - \Delta u + u^3 = 0$ in a domain $\Omega \subset \mathbb{R}^n (n \geq 3)$ is uniquely determined near $\Gamma$ by its Cauchy data on a smooth noncharacteristic surface $\Gamma \subset \Omega$.

Uniqueness follows from the results for linear equations, because subtracting two nonlinear equations (for $u_2$ and $u_1$), we obtain for the difference $u = u_2 - u_1$ the linear hyperbolic equation $\partial_t^2 u - \Delta u + cu = 0$ with $c = u_2^2 + u_2 u_1 + u_1^2$, which depends on $t$. Due to $t$-dependence, Corollary 3.4.3 cannot be applied. On the other hand, uniqueness of the continuation across $\Gamma$ holds for pseudo-convex $\Gamma$ (e.g., as described in Theorem 3.4.1).

There are serious difficulties in trying to generalize Theorem 3.2.1 onto systems of equations, and the main tool is reduction to a system with diagonal principal part and then application of Theorem 3.2.1 or of similar results for scalar differential operators. Even for classical systems of mathematical physics there are challenging open questions.

**Problem 3.5.** Obtain Carleman estimates in the anisotropic case (see Theorem 3.2.1) for not necessarily compactly supported functions (with boundary terms). In particular, the question is interesting for equations of second order.

Most likely this problem can be resolved by combining methods and results of the papers [Is12] and [Tat3].

**Problem 3.6.** By using Carleman estimates obtain uniqueness and stability results for some interesting anisotropic (linear) systems of mathematical physics.

There is some progress in this direction. In particular, the paper of Isakov, Nakamura, and Wang in [I3] handles the isotropic elasticity with residual stress. A particular interesting case is the transversally isotropic elasticity. It is possible that Carleman-type estimates, like (3.5.11), for such systems can be obtained directly, and then one can expect optimal regularity assumptions on coefficients and boundary of the domain. It is also interesting to obtain the Carleman estimate (3.5.11) with boundary terms (i.e. for arbitrary vector-functions $\mathbf{U} \in H_{(2)}(\Omega)$). At present it is not even known that a Carleman estimate for a system implies a Carleman estimate for a (principally) perturbed system.

It seems that the fundamental ideas of Friedrichs [Fri] properly adjusted to the problem can be of great use. In particular, one can try to combine them with the scheme of Hörmander [Hö1] partially outlined after Excercise 3.4.4. We think that a promising direction is to utilize and rethink the results of Lax [Lax] about the Cauchy problem and geometrical optics.

# 4

# Elliptic Equations: Single Boundary Measurements

## 4.0 Results on elliptic boundary value problems

In this chapter we consider the elliptic second-order differential equation

$$Au = f \quad \text{in } \Omega,\ f = f_0 - \sum_{j=1}^{n} \partial_j f_j \tag{4.0.1}$$

with the Dirichlet boundary data

$$u = g_0 \qquad \text{on } \partial\Omega. \tag{4.0.2}$$

We assume that $A = \operatorname{div}(-a\nabla) + b \cdot \nabla + c$ with bounded and measurable coefficients $a$ (symmetric real-valued $(n \times n)$ matrix) and complex-valued $b$ and $c$ in $L_\infty(\Omega)$. Another assumption is that $A$ is an elliptic operator; i.e., there is $\varepsilon_0 > 0$ such that $a(x)\xi \cdot \xi \geq \varepsilon_0 |\xi|^2$ for any vector $\xi \in \mathbb{R}^n$ and any $x \in \Omega$. Unless specified otherwise, we assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$ with the boundary of class $C^2$. However, most of the results are valid for Lipschitz boundaries. We recall that for any $u \in H_{(1)}(\Omega)$ there are boundary values $u$, $\nabla u$ on $\partial\Omega$ that are contained in $H_{(1/2)}(\partial\Omega)$, $H_{(-1/2)}(\partial\Omega)$. These boundary values can be understood via approximations as follows. One can approximate $u$ in $H_{(1)}(\Omega)$ by functions $u_m \in C^\infty(\bar{\Omega})$, and then boundary values are understood as the limits. From trace theorems it follows that $\|u_m\|_{(1/2)}(\partial\Omega)$, $\|\nabla u_m\|_{(-1/2)}(\partial\Omega)$ are bounded by $C\|u_m\|_{(1)}(\Omega)$. In particular, $\|\phi_m \partial_j u_m\|_2(\partial\Omega) \leq C\|\phi_m\|_{(1)}(\Omega)\|u_m\|_{(1)}(\Omega)$. Since approximations are convergent in $H_{(1)}(\Omega)$, they are fundamental in this space, and so limit of their boundary traces is the needed trace of $u$. A (weak) solution $u \in H_{(1)}(\Omega)$ is defined in the sense of the integral identity

$$\int_\Omega (a\nabla u \cdot \nabla\phi + b \cdot \nabla u\phi + cu\phi) - \int_{\partial\Omega} \partial_{\nu(a)} u\phi$$
$$= \int_\Omega \left( f_0\phi + \sum_{j=1}^{n} f_j \partial_j \phi \right) \tag{4.0.3}$$

for any test function $\phi \in H_{(1)}(\Omega)$, provided that the boundary Dirichlet condition (4.0.2) is satisfied and supp $f_j \subset \Omega$.

We will formulate basic results about solvability and regularity of the Dirichlet problem (4.0.1), (4.0.2).

**Theorem 4.1.** *Let $\partial\Omega$ be Lipschitz, $g_0 \in H_{(1/2)}(\partial\Omega)$, and $f = f_0 + \sum_{j\le n} \partial_j f_j$ with $f_j \in L_2(\Omega)$. Let us replace $c$ by $c + \lambda$.*

*Then for all $\lambda \notin E$, where $E$ is a set of points converging to $\infty$, there is a unique generalized solution $u \in H_{(1)}(\Omega)$ to the Dirichlet problem* (4.0.1), (4.0.2). *Moreover,*

$$\|u\|_{(1)}(\Omega) \le C\left(\|g_0\|_{(1/2)}(\partial\Omega) + \sum_{0\le j\le n} \|f_j\|_2(\Omega) + \|u\|_2(\Omega)\right), \tag{4.0.4}$$

*where $C$ depends only on $\Omega$, $\lambda$ and the mentioned bounds on the coefficients of $A$.*

*If a solution is unique, then it does exist. Moreover, one can drop the last term in the previous bound. In particular, if $c \ge 0$, $\Im b = 0$, or if $b = 0$, $\Re c \ge 0$, then a solution exists and is unique.*

*If $f_j \in L_\infty(\Omega_0)$ for some $\Omega_0 \subset \Omega$, then $u \in C^\mu(\Omega_0)$ for some $\mu \in (0,1)$. In addition, when $\bar\Omega_0 \subset \Omega$ and $p > n$, there is constant $C$ depending on the same parameters as above such that*

$$\|u\|_{(1)}(\Omega_{01}) + \|u\|_\infty(\Omega_{01}) \le C\left(\sum_{j=0}^{n} \|f_j\|_p(\Omega_0) + \|u\|_2(\Omega_0)\right).$$

*Here and below $\Omega_{01} \subset \Omega_0$ is any domain with positive distance to $\partial\Omega_0$.*

*If $a, b, c \in C^\gamma(\bar\Omega_0)$, $f_j \in C^\gamma(\bar\Omega_0)$, $g_0 \in C^{1+\gamma}(\partial\Omega_0)$, $\partial\Omega_0 \in C^{2+\gamma}$, then for any domain $\Omega_{01}$ with positive distance to $\partial\Omega_0 \cap \Omega$ there is a constant $C$ depending only on $\Omega_{01}$ and on the norms of $a, b, c$ in $C^\gamma(\Omega_0)$ such that*

$$|u|_{1+\gamma}(\Omega_{01}) \le C\left(\sum_{0\le j\le n} |f_j|_\gamma(\Omega_0) + |g_0|_{1+\gamma}(\partial\Omega_0 \cap \partial\Omega) + |u|_0(\Omega_0)\right). \tag{4.0.5}$$

*If $\nabla a \in L_\infty(\Omega_0)$, $f_1 = \cdots = f_n = 0$ in $\Omega_0$, $f \in L_\infty(\Omega_0)$, $\partial\Omega_0 \in C^2$, $g_0 \in H_{(3/2)}(\partial\Omega_0 \cap \partial\Omega)$, then $u \in H_{(2)}(\Omega_{01})$ for any $\Omega_{01}$ mentioned above.*

*If $f_1 = \cdots = f_n = 0$; $a, \nabla a \in C(\bar\Omega)$; $b, c \in L_\infty(\Omega)$; $f \in L_p(\Omega_0)$, and $g_0 \in C^2(\partial\Omega_0 \cap \partial\Omega)$, then $u \in H_{p;k}(\Omega_{01})$, and there is $C$ depending on $\Omega_{01}$, $p$, the norms of the coefficients in the above-mentioned spaces, and the ellipticity constant of $A$ such that*

$$\|u\|_{2,p}(\Omega_{01}) \le C(\|f\|_p(\Omega_0) + |g_0|_2(\partial\Omega \cap \partial\Omega_0) + \|u\|_2(\Omega_0)).$$

*If $f_1 = \cdots = f_n = 0$; $a, \nabla a, b, c, f \in C^\gamma(\bar\Omega_j \cap \Omega_0)$; $\partial\Omega_j \cap \bar\Omega_0 \in C^2$; $\bar\Omega = \cup\bar\Omega_j$, where $\Omega_j$ are some disjoint subdomains of $\Omega$, then $u \in C^1(\bar\Omega_j \cap \bar\Omega_{01})$. If* *in addition,* $\partial\Omega_j \cap \bar{\Omega}_0 \in C^{2+\gamma}$*, then* $u \in C^{2+\gamma}(\bar{\Omega}_j \cap \bar{\Omega}_{01})$ *and*

$$|u|_{2+\gamma}(\Omega_{01} \cap \bar{\Omega}_j) \\ \leq C\left(\sum_j |f|_\gamma(\Omega_0 \cap \Omega_j) + |g_0|_{2+\gamma}(\partial\Omega \cap \partial\Omega_0) + \|u\|_\infty(\Omega_0)\right),$$

*where C depends on* $\Omega_{01}$*,* $\gamma$*, the norms of coefficients in the above-mentioned Hölder spaces, and the ellipticity constant of A.*

A proof of this result can be found in the book of Ladyzhenskaya and Ural'tseva [LU], pp. 149, 189, 202, 222, except of the bound (4.0.5) and the subsequent $H_{2,p}(\Omega_{01})$-bound, which are obtained by Agmon, Douglis, and Nirenberg [ADN], and the $L_\infty$-bound in the case of divergent equations with measurable and bounded coefficients given by Kinderlehrer and Stampacchia [KinS]. Theorem 4.1 is in fact a formulation of several results of the available theory of elliptic boundary value problems for equations of second order that are sufficient for considering the inverse problems in this book. Of course, it is neither comprehensive nor most general. The domain $\Omega_0$ is needed to formulate local results when the data of the problem are regular only in a subdomain of $\Omega$. When $\Omega = \Omega_0 = \Omega_{01}$, the results are more transparent. Observe that for Lipschitz $\Omega$ a solution to the Dirichlet problem with smooth data can be not in $H_{(2)}(\Omega)$.

**Theorem 4.2** (Comparison Principle). *Assume that* $\Im b = 0$*,* $c \geq 0$ *in* $\Omega$*.*

*If* $f_1 \leq f_2$ *in* $\Omega$ *and* $g_{01} \leq g_{02}$*, then for solutions* $u_1, u_2$ *to the Dirichlet problem* (4.0.1), (4.0.2) *with the data* $f_1, g_{01}$ *and* $f_2, g_{02}$ *we have* $u_1 \leq u_2$*.*

*(Hopf Maximum Principle) If a solution u to equation (4.0.1) with* $f = 0$ *is in* $C(\bar{\Omega})$ *then* $\|u\|_\infty(\Omega) \leq \sup|g_0|$ *over* $\partial\Omega$*. If* $f \geq 0$ *in* $\Omega$*, then* $\inf_\Omega u \geq \inf_{\partial\Omega} g_0$*.*

*(Giraud Maximum Principle) Assume in addition that* $\partial\Omega \in C^{1+\lambda}$ *and that* $a \in C^{1+\lambda}(\bar{\Omega})$*,* $b, c \in C^\lambda(\bar{\Omega})$*. Let l be any nontangential outward direction at* $x \in \partial\Omega$*. Assume that* $c \geq 0$*,* $f \geq 0$ *in* $\Omega$*.*

*If x is a maximum point of* $u \in C(\bar{\Omega})$ *in* $\Omega$ *and* $u(x) > 0$*, then there is* $\varepsilon > 0$ *such that*

$$\varepsilon t < u(x + tl) - u(x)$$

*when* $0 < t < \varepsilon$*. When there is* $\partial_l u(x)$*, then* $\partial_l u(x) > 0$*.*

A proof of the Hopf maximum principle for $a \in C^1(\bar{\Omega})$ and for $u \in C^2(\bar{\Omega})$ can be found in the book of Miranda [Mi], p. 6. In the slightly more general case we consider, it follows by using the approximation results in [LU], p. 158, which claims that if the coefficients of (4.0.1) are convergent almost everywhere, uniformly satisfying boundedness and ellipticity conditions, and if the source terms and the boundary conditions are convergent correspondingly in $H_{(-1)}(\Omega)$ and $H_{(1/2)}(\partial\Omega)$, the solutions to the Dirichlet problem (4.0.1), (4.0.2) are convergent in $H_{(1)}(\Omega)$.

The inequality $u \geq 0$ for functions in $H_{(1)}$ is understood in the sense that such functions can be $H_{(1)}$-approximated by nonnegative smooth functions. For maximum principles for weak solutions we refer also to the book of Kinderlehrer and Stampacchia [KinS], p. 38.

It is easy to see that the Hopf maximum principle implies the comparison principle. To see this, observe that $u = u_2 - u_1$ satisfies equation (4.0.1) with $f = f_2 - f_1 \geq 0$, so by the second part of the Hopf principle we conclude that $u \geq g_{02} - g_{01} \geq 0$.

We are interested in finding $a, b, c$ given the additional boundary data

$$\partial_{\nu(a)} u = g_1 \text{ on } \Gamma, \tag{4.0.6}$$

which is a part of $\partial\Omega$. According to Theorem 4.0.1, if $a, b, c, f$ are $C^1$-smooth near $\partial\Omega \in C^{2+\lambda}$ and $g_0 \in C^{1+\lambda}(\Gamma)$, then $\nabla u$ is continuous near $\partial\Omega$ and has a continuous extension onto $\Omega \cup \Gamma$, so the Neumann data can be understood in the classical sense.

## 4.1 Inverse gravimetry

One is looking for a domain $D$ from its exterior gravitational (Newtonian) potential $u$, which is a solution to the Poisson equation

$$-\Delta u = k\chi(D) \text{ in } \mathbb{R}^3, \qquad \lim_{|x|\to\infty} u(x) = 0. \tag{4.1.1}$$

Let $u_j$ be the gravitational potential that corresponds to a domain $D_j \subset \Omega$. Here $\Omega$ is a known bounded domain. We call a domain $D_j$ *star-shaped* with respect to a point $x$ if any ray originating at $x$ intersects $\partial D_j$ at exactly one point, and we call $D_j$ *convex* in $x_n$ if the intersection of any straight line parallel to the $x_n$-axis with $D_j$ is an interval. The center of gravity of $D$ is the point

$$(\text{vol } D)^{-1} \int_D x dD.$$

**Theorem 4.1.1** (Uniqueness of a domain). *Suppose that either (i) $D_j$ are star-shaped with respect to their centers of gravity or (ii) $D_j$ are convex in $x_n$.*

*If $u_1 = u_2$ (or $\nabla u_1 = \nabla u_2$) on $\partial\Omega$ and $k$ is a positive constant, then $D_1 = D_2$.*

A complete proof of this result is available from the book [Is4], section 3.1. We will give a short proof under the additional assumption that $\partial D_j$ is the graph of a Lipschitz function in the polar coordinates with the origin at the center of gravity of $D_j$. This proof is based on the following lemma.

**Lemma 4.1.2** (Orthogonality Relations). *Let $-\Delta u = f$ near $\bar{\Omega}$, $f \in L_p(\Omega)$, $1 < p$, $f = 0$ outside $\Omega$. Then*

$$\int_\Omega f v = \int_{\partial\Omega} (\partial_\nu v u - \partial_\nu u v)$$

*for any solution $v$ to the equation $-\Delta v = 0$ near $\bar{\Omega}$.*

PROOF. By Theorem 4.1 we have $u \in H_{2,p}(\Omega)$. So we can apply integration by parts to obtain

$$\int_\Omega u(-\Delta v) + \int_{\partial\Omega} u\partial_\nu v - \int_{\partial\Omega} \partial_\nu u v = \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \partial_\nu u v = \int_\Omega f v$$

according to the definition (4.0.3) of a (generalized) solution to the equation $-\Delta u = f$ in $\Omega$. The first term in the above equalities is zero because $v$ is harmonic in $\Omega$. The proof is complete. □

By using Lemma 4.1.2 with particular harmonic functions $v$, one can obtain some information about $f$, like the integral of $f$ over $\Omega$ when $v = 1$.

PARTIAL PROOF OF THEOREM 4.1.1. We will give a proof for star-shaped domains that are defined by Lipschitz functions in the polar-coordinate systems originating at domains centers of gravity.

Assume that we have two domains $D_1$ and $D_2$ with the same potential on $\partial\Omega$. By subtracting equations (4.1.1) for $u_2$ and $u_1$ and letting $u = u_2 - u_1$, we obtain

$$-\Delta u = k(\chi(D_2) - \chi(D_1)).$$

The function $u$ is harmonic outside $\Omega$, zero on its boundary, and goes to zero at infinity. By using the maximum principle we conclude that $u = 0$ outside $\Omega$, and by Lemma 4.1.2 with $f = k(\chi(D_2) - \chi(D_1))$ we obtain

$$\int_{D_1} v = \int_{D_2} v \tag{4.1.2}$$

for any function $v$ harmonic near $\bar{\Omega}$. By letting $v(x)$ be 1 and $x_j$, we conclude that the domains $D_1$, $D_2$ have the same volumes and first moments. Therefore, they have the same centers of gravity. Further on we assume that their center of gravity is the origin.

We make an important observation that if $v$ is harmonic, then so is $x \cdot \nabla v$. Using $x \cdot \nabla v + 3v$ in (4.1.2) instead of $v$ and integrating by parts, we obtain

$$I(v) = \int_{\Gamma_{1e}\cup\Gamma_{1i}} v x \cdot \nu d\Gamma - \int_{\Gamma_{2e}\cup\Gamma_{2i}} v x \cdot \nu d\Gamma = 0, \tag{4.1.3}$$

where $\Gamma_{1e} = \partial D_1 \backslash \overline{D}_2$, $\Gamma_{2e} = \partial D_2 \backslash \overline{D}_1$, $\Gamma_{1i} = \partial D_2 \cap D_1$, $\Gamma_{2i} = \partial D_1 \cap D_2$ are various parts of the boundaries of the $D_j$. Here $\nu$ is the unit normal exterior with respect to $D_1 \backslash \overline{D}_2$ on $\Gamma_{1e} \cup \Gamma_{1i}$ and with respect to $D_2 \backslash \overline{D}_1$ on the remaining parts of the boundaries.

By using the theory of (uniform) harmonic approximation and stability of the Dirichlet problem in the Lipschitz domain $D_1 \cup D_2$ (see, e.g., [Is4], sections 1.7 and 1.8) we can transfer the relation (4.1.3) onto any function $v$ that is harmonic in this union and continuous on its closure. The Dirichlet problem for the Laplace equation in a Lipschitz domain is solvable for any continuous boundary data $g$. If $0 \le g \le 1$, then $0 \le v \le 1$ on $D_1 \cup D_2$ by the maximum principle. Due to star-shapedness we have $x \cdot \nu \ge 0$ on $\Gamma_{2e}$ and on $\Gamma_{1e}$ and $x \cdot \nu \le 0$ on $\Gamma_{2i}$ and on $\Gamma_{1i}$, so from (4.1.3) we obtain

$$0 = I(v) \geq \int_{\Gamma_{1e}} gx \cdot \nu d\Gamma + \int_{\Gamma_{1i}} x \cdot \nu d\Gamma - \int_{\Gamma_{2e}} gx \cdot \nu d\Gamma.$$

We can $L_1(\Gamma_{1e} \cup \Gamma_{2e})$-approximate by such $g$ the function that is 1 on $\Gamma_{1e}$ and 0 on $\Gamma_{2e} \backslash \Gamma_{1e}$. Therefore, we get

$$0 \geq \int_{\Gamma_{1e} \cup \Gamma_{1i}} x \cdot \nu d\Gamma = 3 \int_{D_1 \backslash D_2} 1$$

The last equality follows from Green's formula when we observe that the boundary of $D_1 \backslash \overline{D_2}$ is $\Gamma_{1e} \cup \Gamma_{1i}$. This inequality implies that $D_1 \subset D_2$.

Similarly, $D_2 \subset D_1$. The proof is complete. □

The basic idea of the proof, to use the function $x \cdot \nabla v + 3v$ instead of $v$ in the orthogonality relations (4.1.2) and to prescribe the Dirichlet boundary data as 0 or 1, belongs to P. Novikov [No], who partially implemented it in 1938 for the inverse gravitational problem in the plane case. We call this technique the orthogonality method of Novikov. Later, Prilepko [Pr], [PrOV] and Sretensky contributed to this method, in the 1950s to 1970s.

Lemma 4.1.2 gives necessary and sufficient conditions for potentials $u$ to be zero outside $\Omega$.

**Exercise 4.1.3.** Let $-\Delta u = f$ in $\mathbb{R}^3$, $f \in L_p(\Omega)$, $1 < p$, $f = 0$ outside $\Omega$, and $u(x) \to 0$ when $|x| \to \infty$. Show that the orthogonality relation

$$\int_\Omega fv = 0$$

for any function $v$ which is harmonic near $\overline{\Omega}$ is necessary and sufficient for $u$ to be zero outside $\Omega$.

Using this result, show that the Newtonian potential of the ball $\{|x| < R\}$ of density $f(x) = \psi(|x|)$ is zero if and only if $\int_0^R r^2 \psi(r) dr = 0$.

Another observation is that when $\Omega$ is a ball, we can prescribe $u$ only on a part of $\partial\Omega$.

**Exercise 4.1.4.** Let $\Gamma$ be a nonempty open part of the sphere $\partial\Omega$.

(1) Prove that when the potential of a density $f$ supported in $\Omega$ is zero on $\Gamma$, then $u = 0$ outside $\Omega$.

(2) Prove that if for two potentials $u_1, u_2$ of nonnegative volume densities $f_1, f_2$ supported in $\Omega$ one has $|\nabla u_1| = |\nabla u_2|$ on $\Gamma$, then $u_1 = u_2$ outside $\Omega$.

To solve the part (2) consider the difference $u = u_2 - u_1$, conclude that the equality of absolute values of gradients (of gravity forces) on $\Gamma$ implies by analyticity the equality on $\partial\Omega$ and the homogeneous oblique derivative condition for $u$, then make use of Theorem 4.2 and of behaviour of potentials at infinity.

Uniqueness is not valid without geometrical assumptions or for a positive variable $k$ ([Is4], section 3.4). In particular, there are two (and even a continuum of) simply connected domains of unit density with the same exterior potential. When the density $k$ is not constant but only smooth and positive, one can give two $x_n$-convex domains with the same exterior potentials. However, solutions $D \in C^{1+\lambda}$ are isolated for positive given $k \in C^{\lambda}$.

As observed in Section 2.1, the inverse problem of gravimetry is not well posed by Hadamard, so the following logarithmic-type stability estimate is quite natural.

Let $D_j$ be a star-shaped domain $\{|x| < d_j(\sigma)\}$, where $d_j$ satisfies the following conditions: $0 < \rho < d_j < R - \rho$, $|d_j|_{2+\lambda}(\Sigma) \leq M$. We assume that $\Omega$ is the ball $B(0; R)$.

**Theorem 4.1.5 (Stability Estimate).** *There is a constant $C$ depending only on $\rho$ and $M$ such that if $\||\nabla u_2| - |\nabla u_1|\| < \varepsilon$ on $\partial\Omega$, then $|d_2 - d_1| < C|\ln\varepsilon|^{-1/C}$.*

For a proof we refer to the book [Is4], section 3.6.

Now we discuss a simpler linear inverse source problem.

Let $\Omega$ be a bounded Lipschitz domain. Let us consider the Dirichlet problem

$$Au = f \quad \text{in } \Omega, u \in H_{(2)}(\Omega), \quad u = g_0 \text{ on } \partial\Omega, \tag{4.1.4}$$

where $A = \partial_j(a\partial_j) + c$. Let $\mathfrak{U}$ be the differential operator $\partial_k(\alpha_k\partial_k) + \beta$. We adopt the summation convention over the repeated indices $j, k$ from 1 to $n$.

**Theorem 4.1.6** (Uniqueness of Density). *Let us assume that one of the three conditions* (4.1.5), (4.1.6), (4.1.7) *is satisfied:*

$$A = \mathfrak{U}; \tag{4.1.5}$$

$$a\alpha_j \geq \varepsilon_{jj},$$
$$-(\partial_k(\alpha_k\partial_k a) + \partial_k(a\partial_k\alpha_j) + 2c\alpha_j + 2a\beta)\xi_j^2 + \partial_j\alpha_k\partial_k a\xi_j\xi_k$$
$$\geq \varepsilon_1\xi_1^2 + \cdots + \varepsilon_n\xi_n^2,$$
$$\partial_k(\alpha_k\partial_k c + \alpha\partial_k\beta) + 2c\beta \geq 0; \tag{4.1.6}$$

*where $\varepsilon_{jj}, \varepsilon_j$ are nonnegative numbers with positive sum;*

$$f = \alpha f_1 + f_2, \text{ where } \partial_n f_j = 0, \quad \partial_n\alpha \geq 0 \text{ on } \Omega, \tag{4.1.7}$$

*and $\alpha$ is given.*

*If $f \in L_2(\Omega)$ and*

$$\mathfrak{U}f = 0 \text{ in } \Omega \tag{4.1.8}$$

*in the cases* (4.1.5), (4.1.6), *then $f$ entering the Dirichlet problem is uniquely determined by the additional Neumann data $a\partial_\nu u = g_1$ on $\partial\Omega$.*

*In case* (4.1.7), *$f$ is uniquely identified by the Neumann data if the coefficients of $A$ do not depend on $x_n$, and $c \geq 0$.*

PROOF. Since the problem is linear with respect to $u$, $f$, it is sufficient to show that $u = 0$ on $\Omega$ when $g = h = 0$. From the definition (4.0.3) of weak solutions to the equation (4.1.8) we have $(f, \mathfrak{U}^*\phi)_2(\Omega) = 0$ for any function $\phi \in C_0^\infty(\Omega)$. Since $u \in \mathring{H}_{(2)}(\Omega)$, this equality holds with $\phi = u$. Since $\mathfrak{U}^* = \mathfrak{U}$, we have

$$(Au, \mathfrak{U}u)_2(\Omega) = 0. \tag{4.1.9}$$

When $A = \mathfrak{U}$, we conclude that $Au = 0$ in $\Omega$. Since $A$ is elliptic and $u$ has zero Cauchy data on $\partial\Omega$, we obtain $u = 0$ in $\Omega$, and $f = 0$ on $\Omega$ as well.

We consider condition (4.1.6). Integrating by parts and using zero boundary data for $u$, $\partial_\nu u$, we obtain that the left side of (4.1.9) is

$$\begin{aligned}
&(\partial_j(a\partial_j u), \partial_k(\alpha_k\partial_k u))_2 + (cu, \partial_k(\alpha_k\partial_k u))_2 + (\partial_j(a\partial_j u), \beta u)_2 + (c\beta u, u)_2 \\
&\quad = (\partial_k(a\partial_j u), \partial_j(\alpha_k\partial_k u))_2 - (\partial_k(cu), \alpha_k\partial_k u)_2 - (a\partial_j u, \partial_j(\beta u))_2 + (c\beta u, u)_2 \\
&\quad = (a\alpha_k, \partial_j\partial_k u\partial_j\partial_k u)_2 + ((\partial_k a)\alpha_k, 2^{-1}\partial_k(\partial_j u)^2)_2 + (a\partial_j\alpha_k, 2^{-1}\partial_j(\partial_k u)^2)_2 \\
&\qquad + (\partial_j\alpha_k\partial_k a, \partial_j u\partial_k u)_2 - (c\alpha_k + a\beta, \partial_k u\partial_k u)_2 \\
&\qquad - 2^{-1}(\alpha_k\partial_k c, \partial_k u^2)_2 - 2^{-1}(a\partial_j\beta, \partial_j u^2)_2 + (c\beta, u^2)_2.
\end{aligned}$$

If we integrate by parts once more and interchange notations of some repeated indices, we finally conclude that the left side of (4.1.9) is

$$\begin{aligned}
&(a\alpha_k, \partial_j\partial_k u\partial_j\partial_k u)_2 - 2^{-1}(\partial_k(\alpha_k\partial_k a) + \partial_k(a\partial_k\alpha_j) \\
&\quad + 2c\alpha_j + 2a\beta, \partial_j u\partial_j u)_2 + (\partial_j\alpha_k\partial_k a, \partial_j u\partial_k u)_2 \\
&\quad + (0.5\partial_j(a\partial_j\beta) + 0.5\partial_k(\alpha_k\partial_k c) + c\beta, u^2)_2.
\end{aligned}$$

Since this sum is zero, conditions (4.1.6) imply that $\partial_k u$ is constant for some $k$. This constant is zero because $\partial_k u = 0$ on $\partial\Omega$. Then $u = 0$ because it does not depend on $x_k$ in $\Omega$, and $u = 0$ on $\partial\Omega$. Finally, $f = Au = 0$ on $\Omega$ as well.

The proof under condition (4.1.7) is based on the orthogonality method, and it is given in the book [Is4], Theorem 3.3.1. □

When proving uniqueness in the linear inverse problem, we have to show that the relations $Au = f$ in $\Omega$, $u = \partial_\nu u = 0$ on $\partial\Omega$ imply that $f = 0$. So it does not make any difference whether we use one or another set of Dirichlet data $g_0$. The situation will be completely different for the identification of coefficients, which is discussed in Sections 4.2 and 4.3.

Observe that if $a = -1, c = \beta = 0$, and $\alpha_1 = \cdots = \alpha_{n-1} = 0$, conditions (4.1.6) are simplified to $\alpha_n \leq 0$ and $0 \leq \Delta\alpha_n$. Moreover, one of these functions must be nonzero on $\Omega$. Observe, that condition (4.1.5) is satisfied for any elliptic operator $A$, including the Helmholtz operator, that gives certain conditions for uniqueness of density in absense of maximum principles.

In fact, under minor additional assumptions one can obtain quite strong stability estimates for this problem and even existence results.

**Exercise 4.1.7.** Let $\partial\Omega \in C^4$. Prove the following stability estimate:

$$\|u\|_2(\Omega) \leq C(\|g_0\|_{(4)}(\Omega) + \|g_1\|_{(3)}(\Omega)),$$

where $C$ depends only on $\Omega$ and on the numbers $\varepsilon_{jj}, \varepsilon_j$. When $A = \mathfrak{U}$, obtain the better (and optimal) estimate

$$\|f\|_2(\Omega) \leq C(\|g_0\|_{(3/2)}(\partial\Omega) + \|g_1\|_{(1/2)}(\partial\Omega)).$$

{*Hint*: By using extension theorems find a function $v$ bounded by the given norms of the boundary value data for $u$ that has the same boundary data as $u$, subtract $v$ from $u$ and repeat the uniqueness proof with $(Au, Av)_2(\Omega)$ in the right side of (4.1.9) instead of zero, and make use of the Schwarz inequality.}

Existence and stability of the solution $(u, f)$ to the inverse source problem in case (4.1.5), (4.1.8) follows from the relations

$$A^2 u = 0 \text{ in } \Omega, \qquad u = g_0, \partial_\nu u = g_1 \text{ on } \partial\Omega,$$

which constitute the first boundary value problem for the fourth-order elliptic equation. This problem is known to be Fredholm, so uniqueness of its solution implies its existence and Lipschitz stability estimate in both the Hölder and Sobolev classes. We refer to the book of Morrey [Mor], Theorems 6.5.3, 6.4.8.

The elliptic equation $A(\mathfrak{U} - \varepsilon\Delta)u = 0$ (with small positive $\varepsilon$) can be used for finding an approximation $f_\varepsilon$ of a solution $f$ when some of the $\varepsilon_{jj}$ are zero.

## 4.2 Reconstruction of lower-order terms

Now we give an example of an inverse problem that is well-posed. We consider equation (4.0.1) with coefficients that do not depend on $x_n$ in the domain $\Omega = G \times (-H, 0)$, where $G$ is a bounded domain in $\mathbb{R}^{n-1}$ with $C^{2+\lambda}$-boundary $(0 < \lambda < 1)$ and $H$ is a positive number. We assume that $Au = A'u - \partial^2 u/\partial x_n^2$, where $A'$ does not involve derivatives with respect to $x_n$.

We let $\Gamma = G \times \{0\}$, and we are looking for the coefficient $c \geq 0$ of equation (4.0.1) whose solution $u$ satisfies the Dirichlet boundary data (4.0.2) and the additional Neumann data (4.0.6). Let $u_j$ be a solution to the Dirichlet problem (4.0.1), (4.0.2) with $c = c_j$ and $g_{1,j}$ the corresponding Neumann data (4.0.6), $j = 1, 2$.

We assume that $g_0 \in C^{2+\lambda}(\partial\Omega), 0 \leq g_0$, on $\partial\Omega$; $g_0 = 0$ on $G \times \{-H\}$; $\partial_n g_0 \geq 0$ and $\partial^2 g_0/\partial x_n^2 = 0$ on $\partial G \times \{-H\}$; $0 \leq A' g_0$ on $\Gamma$; and $Ag_0 = 0$ on $\partial\Gamma$. About the coefficients of $A$ we assume that $a, b \in C^{1+\lambda}(\overline{\Omega})$; $c \in C^{\lambda}(\overline{\Omega})$.

Observe that the elliptic theory briefly described by Theorem 4.1 guarantees uniqueness and existence of a solution $u \in C^{2+\lambda}(\overline{\Omega})$. Indeed, we have a unique solution $u \in H_{(1)}(\Omega)$ that is in $C^{2+\lambda}(\overline{\Omega}\backslash V)$, where $V$ is some neighborhood of the corner points $\partial\Gamma \times \{0, -H\}$. To show that $u$ is $C^{2+\lambda}$-smooth in $\overline{\Omega}$ near $\Gamma \times \{-H\}$ one can use the even extension with respect to $(x_n + H)$, obtaining an $H_{(1)}$-solution in the extended domain $\Omega$, and then apply the local regularity claim of Theorem 4.1. To show smoothness near $\Gamma \times \{0\}$ one can solve the Dirichlet problem for (smoothly extended) $A$ in a $C^{2+\lambda}$-domain containing $\overline{\Omega}$, with $C^{2+\lambda}$-extended data $g_0$; subtract this $C^{2+\lambda}(\overline{\Omega})$-solution from $u$; and apply to the difference the odd extension with respect to $x_n$ together with the previous argument.

**Theorem 4.2.1.** *Under the given assumptions on $g_0$ the following stability estimate holds:*

$$|c_2 - c_1|_\lambda(G) \leq C|g_{1,2} - g_{1,1}|_{1+\lambda}(\Gamma), \tag{4.2.1}$$

*where $C$ depends only on $|c_j|_\lambda(\overline{G})$, $|a|_{1+\lambda}(\overline{G})$, $|b|_\lambda(\overline{G})$, $|g_0|_{2+\lambda}(\partial\Omega)$, and on the ellipticity constant of $A$.*

In particular, we have uniqueness of recovery of $c$ from the additional Neumann data.

This result follows from uniqueness and stability in the inverse source problem

$$(A + c_2)u = \alpha f, \quad \partial_n f = 0 \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega,$$
$$\partial_\nu u = g_1 \text{ on } \Gamma, \tag{4.2.2}$$

where $\alpha = u_1$ and $f = c_1 - c_2$. The relations (4.2.2) are the results of subtraction of the equations for $u_1$ from the equations for $u_2$. It is interesting to observe that now $\alpha$ depends on the boundary data $g_0$ for the direct problem, so unlike the inverse source problem with fixed $\alpha$, by changing $g_0$ we obtain more information about $f$. This observation will be essential in Chapter 5, which is devoted to many boundary measurements.

The idea of the proof of (4.2.1) given in the paper of Khaidarov [Kh1] is first to establish Schauder-type estimates for the inverse source problem (4.2.2). This can be done by "freezing coefficients," differentiating the equation with respect to $x_n$, and applying standard elliptic estimates. We explain this idea referring for details to the cited paper.

Let $x_0 \in \Gamma$ and $\varepsilon$ small and positive. We write equation (4.2.2) in $\Omega \cap B(x_0; 2\varepsilon)$ as $\alpha^{-1} A_0 u = \alpha^{-1}(A_0 - (A + c_2))u + f$, where $A_0$ is the operator $A$ with the coefficients at the point $x_0$. Differentiating this equation gives the following boundary value problem:

$$\partial_n(\alpha^{-1} A_0 u) = \partial_n(\alpha^{-1}(A_0 - A - c_2))u \text{ in } \Omega \cap B(x_0; 2\varepsilon), \quad \partial_n u$$
$$= g_1 \text{ on } \Omega \cap B(x_0; 2\varepsilon).$$

Schauder-type estimates of Theorem 4.1 for equations in variational form give

$$|\partial_n u|_{1+\lambda}(\Omega \cap B(x_0; \varepsilon)) \leq C(|\alpha^{-1}(A_0 - A)u|_\lambda(\Omega \cap B(x_0; 2\varepsilon)) + |g_1|_{1+\lambda}(\Gamma)$$
$$+ |u|_{1+\lambda}(\Omega \cap B(x_0; 2\varepsilon))$$
$$\leq C(\varepsilon^\lambda |u|_{2+\lambda}(\Omega) + |g_1|_{1+\lambda}(\Gamma) + C(\varepsilon)|u|_0(\Omega)).$$

Using this estimate and the Dirichlet boundary condition on $\Gamma$ from relation (4.2.2) on $\Gamma$, we then obtain that $|f|_\lambda(\Gamma \cap B(x_0; \varepsilon))$ is bounded by the right side of this inequality. Since this is true for any $x_0 \in \Gamma$ and does not depend on $x_n$ on the whole of $\Omega$ we bound the norm of $f$ on $\Gamma$. Hence the Schauder-type estimates for the problem (4.2.2) bound by the same quantity the norm $|u|_{2+\lambda}(\Omega)$. Choosing $\varepsilon$ small we eliminate the term with $|u|_{2+\lambda}(\Omega)$ on the right side and obtain the

Schauder-type estimate $|u|_{2+\lambda}(\Omega) \le C(|g_1|_{1+\lambda}(\Omega) + |u|_0(\Omega))$. The term $|u|_0(\Omega)$ can be eliminated by using compactness-uniqueness arguments as in the proof of Theorem 3.4.11, provided that we have uniqueness of a solution.

Under the assumptions that the coefficients of $A$ are $x_n$-independent and the weight function $\alpha$ is monotone with respect to $x_n$, one proves uniqueness in the inverse source problem by a modification of Novikov's orthogonality method, which we describe briefly below. We will set $\Gamma_H = \Gamma \times \{-H\}$, $\Gamma_0 = \partial\Omega \backslash (\Gamma \cup \Gamma_H)$.

**Lemma 4.2.2.** *If the coefficients of $A$ do not depend on $x_n$, and $\partial_n \alpha > 0$ on $\Omega$, then $f = 0$ on $\Omega$.*

PROOF. We multiply equation (4.2.2) by $v \in H_{(2)}(\Omega)$ and apply Green's formula. We obtain

$$\int_{\Gamma \cup \Gamma_0 \cup \Gamma_H} (\partial_{\nu(A)} v u - v \partial_{\nu(A)} u) \quad d\Gamma + \int_\Omega u A^* v = \int_\Omega \alpha f v.$$

If $A^* v = 0$ on $\Omega$ and $v = 0$ on $\Gamma_0$, then using the boundary conditions (4.2.2) where $h = 0$, we yield

$$\int_\Omega \alpha f v = - \int_{\Gamma_H} v \partial_\nu u.$$

By using the odd extension with respect to $x_n$ we conclude that finite differences (which satisfy the same equation as $v$ because the coefficients are $x_n$-independent) of $v$ with respect to $x_n$ can be used instead of $v$. Passing to the limit in finite differences, we conclude that we can replace $v$ by $\partial_n v$. When in addition, $\partial_n v = 0$ on $\Gamma_H$, we conclude that the integral of $\alpha f \partial_n v$ over $\Omega$ is zero. Integrating by parts with respect to $x_n$, we obtain

$$\int_\Gamma v \alpha f - \int_\Omega v f \partial_n \alpha = 0. \tag{4.2.3}$$

Intending to show as in the proof of Theorems 4.1.1 and 4.1.6 that the left side is positive for some $v$, we introduce the set $\Omega_+ = \{f > 0\}$ and the set $\Omega_- = \{f < 0\}$. Since $f$ does not depend on $x_n$, these sets are cylinders $\Gamma_+ \times (-H, 0)$, $\Gamma_- \times (-H, 0)$. We can assume that both $\Omega_+$ and $\Omega_-$ have nonvoid interiors; otherwise, $\alpha f$ is nonnegative (or nonpositive), and we will obtain a contradiction by using Giraud's maximum principle on $\Gamma$.

Let $v$ be a solution to the mixed problem

$$A^* v = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad \partial_\nu v = 0 \text{ on } \Gamma_H,$$
$$v = g_0 \text{ on } \Gamma.$$

Let us assume that $g_0 \in C_0^2(\Gamma)$, $0 \le g_0 \le 1$, on $\Gamma$, and $g_0 = 0$ on an open subset $\Gamma_{0-}$ of $\Gamma_-$ that does not depend on $g_0$. By using comparison with a fixed solution to the same boundary value problem one can show that $v(x) \le 1 - \varepsilon(x)$, where $\varepsilon$ is a positive continuous function on $\Omega$ that does not depend on $g_0$. Indeed, let $v_0$ be the solution to the above mixed problem for the operator $A^*$ in $\Omega$ with boundary data $g_{00} \in C^2(\Gamma)$ that are 1 outside $\Gamma_{0-}$ and 0 on an open nonvoid subset of $\Gamma_{0-}$. As shown above, this solution exists and is in $C^1(\bar{\Omega})$. By the maximum principles, $0 < v_0(x) < 1$ on $\Omega$. Again by maximum principles (in the form of comparison), $v \leq v_0$, so we can let $\varepsilon(x) = 1 - v_0(x)$. By maximum principles we also have $0 \leq v$ on $\Omega$. Since $\partial_n \alpha > 0$ on $\Omega$, the left side of (4.2.3) is not less than

$$\int_\Gamma g\alpha f - \int_{\Omega_+} (1-\varepsilon) f \partial_n \alpha.$$

We can approximate in $L_1(\Gamma)$ by such $g_0$ the function that is 1 on $\Gamma_+$ and zero on $\Gamma \backslash \Gamma_+$, and conclude that

$$0 \geq \int_{\Gamma_+} \alpha f - \int_{\Omega_+} f \partial_n \alpha + \int_{\Omega_+} \varepsilon f \partial_n \alpha = \int_{\Omega_+} \varepsilon f \ \partial_n \alpha > 0.$$

In the last equality we used that the integral over $\Gamma_+$ combined with the first integral over $\Omega_+$ is zero because $\alpha = u_1 = 0$ on $\Gamma_H$ due to the Dirichlet boundary condition. The contradiction shows that the initial assumption was wrong, and so $f = 0$.

The proof is complete. □

Theorem 4.2.1 will follow from Schauder estimates and from Lemma 4.2.2 if we show that $\partial_n \alpha = \partial_n u_1 > 0$ on $\Omega$. Differentiating the equation for $u_1$ with respect to $x_n$ and using that the coefficients of $A$ and the coefficient $c_1$ do not depend on $x_n$, we obtain the equation $(A + c_1)\partial_n u_1 = 0$ in $\Omega$. Differentiating the boundary condition on $\partial G \times (-H, 0)$ gives $\partial_n u_1 = \partial_n g_0 \geq 0$ there. On $G \times \{-H\}$ we have from the differential equation,

$$\partial_n(\partial_n u_1) = A' u_1 + c_1 u_1 = A' g_0 + c_1 g_0 = 0,$$

and similarly, on $G \times \{0\}$, $\partial_n(\partial_n u_1) \geq 0$, due to the conditions on $g$ and $c_1$. By applying maximum principles, we conclude that $\partial_n u_1 > 0$ in $\Omega$. Indeed, $\partial_n u_1$ attains its minimum on $\bar{\Omega}$. If this minimum is negative, then it cannot be achieved either inside $\Omega$ or on $G \times \{-H\}$, $\partial G \times (-H, 0)$. So it must be achieved at some point $G \times \{0\}$. At this point, by Giraud's maximum principle (Theorem 4.2), we have $\partial_n(\partial_n u_1) < 0$, which contradicts the properties of this function.

A similar problem for parabolic equations (with final overdetermination) is discussed in more detail in Section 9.1, where there are some existence results and monotone iterative algorithms suitable also for numerical solution of the inverse problem. Similar results can be obtained for the elliptic inverse problem under consideration.

In this problem the unknown term does not depend on one of the space variables, which is not quite natural in many applications.

If one has additional boundary data on $G \times \{-H\}$, it is possible to recover in a stable way two terms (e.g., $a$ and the right side $f$, which do not depend on $x_n$). For proofs and further information we refer to the paper of Khaidarov [Kh1].

When $c$ (and other coefficients of $A$) does not depend on $x_1$, and $\partial G \cap V \subset \{x_1 = 0\}$, where $V$ is a ball centered at a point of $\partial G$, then uniqueness of $c$ on the set

$\{x \in \Omega : (0, x_2, \ldots, x_{n-1}) \in V\}$ can be obtained by the method of Carleman-type estimates originated by Bukhgeim and Klibanov [BuK]. The essential condition is that $g_0 \geq 0$ on $\partial\Omega$ and $> 0$ on $\Gamma \cap V$. This inverse problem is not well-posed, in contrast to the problem with $x_n$-independent $c$. One can obtain only conditional Hölder-type stability estimates as in the Cauchy problem for elliptic equations. Due to space limitations we will not discuss this here in detail, referring to the review paper of Klibanov [K1] and to the recent book [KlT]. For hyperbolic equations the method of Carleman estimates is demonstrated in Section 8.2.

Identification of the coefficient $c$ is of interest for theory of semiconductor devices. We refer to more detail to the paper of Burger, Engl, Markovich, and Pietra [BuEMP]. One of accepted mathematical models for semiconductors is represented by a system of three quasilinear elliptic partial differential equations of second order. In particular, it is of interest [BuEMP], p.1777, to identify the so-called doping profile $c_* \in L_\infty(\Omega)$ from the elliptic equation $div(a\nabla v) = 0$ in $\Omega$ with $c_* = -\Delta log a + a - a^{-1}$. The well known substitution $v = a^{-1/2}u$ transforms the equation for $v$ into $-\Delta u + cu = 0$ with $c = a^{-1/2}\Delta a^{1/2}$. The additional data for determining $c$ can be few sets of the Cauchy for $u$ or all possible Cauchy data on a part of $\partial\Omega$.

The coefficient $c$ which is a function of all space variables can not be uniquely determined from few sets of the boundary data. The assumiption that $c$ does not depend on one of space variables leads to a nice theory described above, but it is artificial in many applications. Another option ( piecewise constant doping profiles of importance for semiconductors theory) is to consider the equation

$$-\Delta u + k\chi(D)u = 0 \text{ in } \Omega, \tag{4.2.4}$$

where a domain $D$ is to be found from few sets of the boundary Cauchy data for $u$. Global uniqueness results for this inverse problem are still not available. Linearizations around $k = 0$ lead to the inverse source problem with the unknown source term $u_0\chi(D)$ where $u_0$ is the harmonic function, $u_0 = u$ on $\partial\Omega$. For an appropriate choice of the Dirichlet data for $u$ one can derive global uniqueness results for $D$ within star-shaped or $x_n$-convex domains. When $D$ is close to a fixed domain $D_0$, one can obtain uniqueness results by using a device from the theory of variational inequalities like in [Is4], Corollary 5.1.4.

Turning to inverse problems for nonlinear elliptic equations

$$-\Delta u + c(u) = 0 \text{ in } \Omega, \tag{4.2.5}$$

we observe that at present there are only local uniqueness results for (small) $c$ when in addition to the Dirichlet data $g_0$ we prescribe the Neumann data on $\partial\Omega$ as well. We refer to the paper of Pilant and Rundell [PiR2], where an essential assumption is that $g_0$ is in a certain sense monotone on $\partial\Omega$.

However, there is an important inverse problem for equation (4.2.5) originating in magnetohydrodynamics where $u$ is constant on $\partial\Omega$. Equation (4.2.5) with these boundary data describes the plasma in equilibrium. This inverse problem was studied by Beretta and Vogelius [BerVo]. For natural configurations ($\Omega$ is a ball or a torus) there are examples of nonuniqueness. Indeed, when $\Omega$ is a ball known results for nonlinear elliptic equations imply that $u$ depends only on the distance to the center of the ball. In resulting ordinary differential equation (in polar coordinates) one can not find a function $c(u)$ from one additional number ( the derivative of solution at the endpoint) even assuming that $c$ is analytic. Uniqueness is shown in [BerVo] for analytic $c$ when $\Omega$ has an analytic corner whose interior angle is not $\pi/2$ or $\pi$. Their proof consists in demonstrating that compatibility conditions and additional Neumann data determine all derivatives of $c$ at a corner point. For recent results about this problem and for another approach based on complex variables we refer to Demidov and Moussaoui [DM].

## 4.3 The inverse conductivity problem

Of special interest is the so-called conductivity equation

(4.3.1) $$\operatorname{div}(a\nabla u) = 0, \text{ in } \Omega, \; a \in L_\infty(\Omega).$$

First we give a simple global uniqueness result due to Alessandrini [Al2]. Let $u_1, u_2$ be solutions to the Dirichlet problem (4.3.1), (4.0.2) with scalar $a = a_1$, $a = a_2$, which are equal to 1 near $\partial\Omega$ and nonconstant $g$. If $a_1 \leq a_2$ in $\Omega$ and $\partial_\nu u_1 = \partial_\nu u_2$ on $\Gamma$, then $a_1 = a_2$ on $\Omega$. Here $\Gamma$ is any nonvoid open part of $\partial\Omega$.

In fact, $u_1, u_2$ are harmonic near $\partial\Omega$, and they have the same Cauchy data on $\Gamma$. Therefore, they coincide near $\partial\Omega$. By the definition (4.0.3) of a weak solution,

$$\int_\Omega a_1|\nabla u_1|^2 = \int_{\partial\Omega} u_1\partial_\nu u_1 = \int_{\partial\Omega} u_2\partial_\nu u_2 = \int_\Omega a_2|\nabla u_2|^2 \geq \int_\Omega a_1|\nabla u_2|^2.$$

By the Dirichlet principle, $u_1$ is the unique minimizer of the first integral (the energy integral), so we conclude that $u_1 = u_2$ on $\Omega$, and therefore $a_1 = a_2$.

Now we consider

(4.3.2) $$a = 1 + k\chi(D), \quad \bar{D} \subset \Omega$$

where $D$ is a subdomain of $\Omega$ to be found with $C^1$-piecewise smooth boundary. Unless specified, in this section we assume that $k$ is a given nonzero constant.

We discuss the direct problem in more detail. By Theorem 4.1, if $g_0 \in C^{2+\lambda}(\partial\Omega)$ and $\partial\Omega \in C^{2+\lambda}$, then a solution $u$ to the Dirichlet problem (4.0.1), (4.0.2) belongs to $C^{2+\lambda}(\overline{\Omega} \cap V)$, where $V$ is a neighborhood of $\partial\Omega$. Denote $u$ on $\Omega\backslash\overline{D}$ by $u^e$ and $u$ on $D$ by $u^i$. When $\partial D$ is only Lipschitz, then $u$ is continuous in $\bar{\Omega}$ and $\nabla u^e(\nabla u^i)$ has nontangential limits at almost every point of $\partial D$. Moreover, these limits belong to $L_2(\partial D)$, and for any sequence of $\nu$-approximating $D$ domains $D_{m+1} \subset D_m$, we have $\|\nabla u^e\|_2(\partial D_m) < C$ (property $B$). These results are contained or directly follow from the paper of Escauriaza, Fabes, and Verchota [EsFV]. $\nu$-approximation means that in a local coordinate system where $D$ is given by the inequality $\{x_n < d(x')\}$ with a Lipschitz $d$, domains $D_m$ are defined as $\{x_n < f_m(x')\}$ where $f_m$ satisfy the conditions: $d + (m+1)^{-1} < f_m \leq d + m^{-1}$, $\nabla f_m \to \nabla d$ almost everywhere and $|\nabla f_m| \leq C$. The estimate on the boundaries of approximating domains follows from the estimate of the maximal functions in the paper [EsFV]. When $\partial D \in C^{1+\lambda}$, the classical method of simple layer potentials described in the book of Miranda [Mi] (compare with the proof of Lemma 4.3.8) gives solutions with $\nabla u^e(\nabla u^i)$ continuous up to $\partial D$. When $\partial D \in C^{2+\lambda}$, then by Theorem 4.1 $u^e \in C^{2+\lambda}(\Omega \backslash D)$ and $u^i \in C^{2+\lambda}(\overline{D})$. Equation (4.3.1) is then equivalent to the following equations and refraction conditions on $\partial D$:

$$\Delta u^e = 0 \quad \text{on } \Omega\backslash\overline{D}, \qquad \Delta u^i = 0 \text{ on } D,$$
$$u^e = u^i, \quad \partial_\nu u^e = (1+k)\partial_\nu u^i \quad \text{on } \partial D. \tag{4.3.3}$$

**Exercise 4.3.1.** Let $u \in H_{(1)}(\Omega)$, $\nabla u^e$ have nontangential limits almost everywhere on $\partial D$ and possess the property $B$, and so are $\nabla u^i$. Prove that the differential equations (4.3.3) and the refraction conditions are equivalent to the differential equation (4.3.1) with the coefficient $a$ given by (4.3.2).

{*Hint*: In the definition (4.0.3) of a solution to equation (4.3.1) first consider test functions $\phi$ in $C_0^\infty(D)$ and in $C_0^\infty(\Omega\backslash\overline{D})$ and conclude that the differential equations are satisfied; then integrate by parts in this definition in $D$ and $\Omega\backslash\overline{D}$, transferring derivatives onto $u^e$, $u^i$, and use the density of the test functions in $L^2(\partial D)$. When integrating by parts in Lipschitz $D$ use $\nu$-approximations, integrate in the approximating domains, and pass to the limit with respect to these domains.}

Domains $D_1, D_2$ are called $i$-contact domains if the sets $\Omega\backslash\overline{D_j}$, $\Omega\backslash(\overline{D_1} \cup \overline{D_2})$, $D_1 \cap D_2$ are connected; the sets $\partial D_1 \cap \partial D_2$, $\text{int}(\overline{D_1} \cap \overline{D_2})$ are disjoint, and there is a nonempty $C^2$-surface that belongs to the boundaries of both $\Omega\backslash(\overline{D_1} \cup \overline{D_2})$ and $D_1 \cap D_2$. In other words, these domains have a common piece of the boundaries. They are located on "one side" of this piece, and they satisfy some natural topological restrictions. In particular, two domains are $i$-contact when they are $x_n$-convex, their boundaries have a common part, and these domains overlap.

**Theorem 4.3.2.** *Let $D_1$, $D_2$ be $i$-contact Lipschitz solutions to the inverse conductivity problem* (4.3.1), (4.3.2), (4.0.2), (4.0.6) *with nonconstant Dirichlet data.*

*Then $D_1 = D_2$.*

PROOF. Let $\Gamma_0$ be the common piece of $\partial D_1$ and $\partial D_2$. The functions $u_1^e$ and $u_2^e$ are harmonic on $\Omega\backslash(\overline{D_1} \cup \overline{D_2})$, and they have the same Cauchy data on $\Gamma$. By uniqueness in the Cauchy problem they coincide there. Using (4.3.3) we conclude that $u_1^i = u_2^i$ and $\partial_\nu u_1^i = \partial_\nu u_2^i$ on $\Gamma_0$. Again by uniqueness in the Cauchy problem, $u_1^i = u_2^i$ on $D_1 \cap D_2$.

Let us assume that $D_1\backslash D_2$ is not empty. The functions $u_1^i$ and $u_2^e$ are harmonic in $D_1\backslash\overline{D_2}$. Due to our assumptions on $D_1$, $D_2$, the boundary of $D_1\backslash\overline{D_2}$ consists only of points of $\partial(D_1 \cup D_2)$ or of $\partial(D_1 \cap D_2)$. In the first case, by using the refraction conditions (4.3.3), we have $u_1^i = u_1^e = u_2^e$, and in the second case, $u_1^i = u_2^i = u_2^e$. So the functions $u_1^i$ and $u_2^e$ are harmonic in $D_1\backslash\overline{D_2}$ and coincide on its boundary. By the maximum principles they are equal in $D_1\backslash D_2$. Therefore, they have the same normal derivatives on $\partial(D_1\backslash\overline{D_2})$.

We consider $\partial D_1 \backslash \overline{D_2}$. On this set we have $\partial_\nu u_1^i = \partial_\nu u_2^e = \partial_\nu u_1^e$ because $u_1^e = u_2^e$ near this set. On the other hand, the refraction conditions give $(1+k)\partial_\nu u_1^i = \partial_\nu u_1^e$ on the same set, which implies that $\partial_\nu u_1^i = 0$ on $\partial D_1 \backslash \overline{D_2}$. Similarly, $\partial_\nu u_1^i = 0$ on the remaining part of $\partial(D_1 \backslash \overline{D_2})$. By uniqueness in the Neumann problem, $u_1^i$ is constant, and therefore $u_2^e$ is constant, which contradicts our assumptions about the Dirichlet data for $u$.

This contradiction shows that $D_1 \subset D_2$. Similarly, $D_2 \subset D_1$.

The proof is complete. □

**Exercise 4.3.3.** Prove that if $\partial_\nu u_1^i = 0$ on $\partial(D_1 \backslash \overline{D_2})$, then $u_1^i$ is constant on $D_1 \backslash D_2$.

{*Hint*: make use of integration by parts and of $\nu$-approximation of $D_1 \backslash \overline{D}_2$ by their open subsets with smooth boundaries.}

**Exercise 4.3.4.** Let $\Omega$ be the cylinder $G \times (-H, 0)$, where $G$ is a bounded $C^{2+\lambda}$-domain in $\mathbb{R}^{n-1}$. Let $D$ be its subdomain $\{-H < x_n < d(x'), x' \in G\}$, where $d$ is a Lipschitz function, $-H < d < 0$. Let $\Gamma$ be the union of two nonempty open parts of $G \times \{-H\}$ and of $G \times \{0\}$.

Show that $D$ is uniquely identified by the data of the inverse problem (4.3.1), (4.3.2), (4.0.2), (4.0.6), provided that $g_0 = 0$ on $\partial G \times (-H, 0)$.

We observe that these two exercises are interesting for $D$ with piecewise smooth and even with $C^{2+\lambda}$-boundaries, and proofs for general Lipschitz domains are only slightly more difficult, because one has to use some approximation.

Now we turn to noncontact domains. Not as much is known here. We are able to give only the following global uniqueness result.

**Theorem 4.3.5.** *Let us assume that $D_1$, $D_2$ are either (i) two convex polyhedra or (ii) two bounded cylinders with strictly convex bases. Let*

$$\operatorname{diam} D_j < \operatorname{dist}(D_j, \partial\Omega). \tag{4.3.4}$$

*If the solutions $u_j$ of* (4.3.1), (4.3.2), (4.0.2) *with nonconstant $g_0$ and with $D = D_j$ satisfy the condition $\partial_\nu u_1 = \partial_\nu u_2$ on $\Gamma$, then $D_1 = D_2$.*

It is observed in the paper of Friedman and Isakov [FrI] that condition (4.3.4) can be dropped when $\Omega$ is a half-space. Theorems 4.3.2, 4.3.5 are from [FrI].

We mention that global uniqueness is known for unions of a finite numbers of discs in the plane and for balls $D$ in the three-dimensional space (see the papers of Isakov and Powell [IsP] and of Kang and Seo [KS1], [KS2]), but it is not known for ellipses or ellipsoids.

*Outline of a proof for $n = 2$.* First we prove the following Lemma.

**Lemma 4.3.6.** *Let the origin be a vertex of a convex polygon $D$ and let $u^e$ have a harmonic continuation onto a ball $B(0; \varepsilon)$. Then there is a rotation of the plane such that $u^e$ on this ball is invariant with respect to this rotation.*

PROOF. We can assume that $D_\varepsilon = D \cap B(0;\varepsilon)$ is $\{0 < \sigma < \theta, 0 < r < \varepsilon\}$. According to relations (4.3.3), the function $u^i$ solves the following Cauchy problem: $\Delta u^i = 0$ in $D_\varepsilon$, $u^i = u^e$, $\partial_\nu u^i = (1+k)^{-1}\partial_\nu u^e$ on $\{0 < x_1 < \varepsilon, x_2 = 0\}$. We can consider the same extended Cauchy problem with the data on the interval $\{-\varepsilon < x_1 < \varepsilon\}$. Since its data $u^e$ are analytic near the origin, for some $\varepsilon$ there is a solution $u^i$ that is analytic on $B(0;\varepsilon)$ (e.g., see the book of John [Jo4]). A solution is unique on $D_\varepsilon$, so $u^i$ has an analytic continuation onto $B(0;\varepsilon)$ for some positive $\varepsilon$.

The functions $u^i, u^e$ are harmonic near the origin, so they admit the expansions $u(x) = \sum(a_m \cos m\sigma + b_m \sin m\sigma)r^m (m = 0, 1, \dots)$ with the coefficients $a^i_m, a^e_m, \dots$ for $u^i, u^e$. From the refraction conditions (4.3.3) at $\sigma = 0$ we obtain $a^e_m = a^i_m$ and $b^e_m = (1+k)b^i_m$. The same conditions at $\sigma = \theta$ then give $(b^e_m - b^i_m)\sin m_\theta = 0$ and $(a^e_m - (1+k)a^i_m)\sin m\theta = 0$. Summing up, we obtain $b^e_m \sin m\theta = a^e_m \sin m\theta = 0$.

If $\theta/\pi$ is irrational, then $b^e_m = a^e_m = 0$ for all $m = 1, \dots$. Therefore, the function $u^e$ is constant, which contradicts the assumption that the Dirichlet data $g$ are not constant. If $\theta/\pi$ is $p/q, q \geq 1$ and $(p, q) = 1$, then for all nonzero coefficients we must have $\sin mp\pi/q = 0$, which means that $mp/q$ is an integer; so $m = lq, l = \dots, -1, 0, 1, \dots$. Returning to the representation of $u^e$ by the series, we conclude that this function is invariant with respect to rotation by $2\pi/q$.

The proof is complete. □

We return to the proof of Theorem 4.3.5. Let us assume that $D_1$ is not $D_2$. Then we may assume that the origin is a vertex of $D_1$ which is not contained in the convex hull of $D_1 \cup D_2$. Since $u^e_1, u^e_2$ are harmonic on the domain $\Omega\backslash(\overline{D_1} \cup \overline{D_2})$ and have the same Cauchy data on $\Gamma$, they are equal on this domain; and since $u^e_2$ has a harmonic continuation onto a neighborhood of the origin, $u^e_1$ has a harmonic continuation as well. By condition (4.3.4) there is a sector $S$ of the disk $B(0; R)$, where $R > \operatorname{diam} D_1$, with angle greater than $\pi$ that is contained in $\Omega\backslash\overline{D_1}$. Rotating this sector $q$ times by the angle $2\pi/q$, one can obtain the harmonic continuation of $u^e_1$ onto the whole disk that contains $\overline{D_1}$ and therefore onto $\Omega$. Indeed, let $O$ be this rotation. Then $u(O(x)) = u(x)$ when $|x| < \varepsilon$ by Lemma 4.3.6. Since both functions are harmonic, they agree also on the intersection of $S$ and $O(S)$, so $u(O(x))$ is the harmonic continuation of $u$ onto $O(S)$, and we can proceed with rotations. The functions $u^e_1$ and $u^i_1$ are harmonic in $D_1$, and they coincide on its boundary. Therefore, they are equal in $D_1$. Using the second refraction condition (4.3.3), we conclude that $\partial_\nu u^e_1 = 0$ on $\partial D_1$. By uniqueness in the Neumann problem for harmonic functions, $u^e_1$ is constant on $D_1$ , and by uniqueness of the harmonic continuation, it is constant on $\Omega$, which contradicts the assumption that the Dirichlet data $g_0$ are not constant. So the initial assumption that $D_1$ is not $D_2$ is wrong, and the proof is complete.

The proof for three-dimensional space and when $\Omega$ is a half-space is given for polyhedra in the paper of Friedman and Isakov [FrI] and for cylindrical $D$ in the paper of Isakov and Powell [IsP]. Recently, Seo in the paper [Se] showed that two boundary measurements uniquely determine a (not necessarily convex) polygon $D$

and $k$. We observe that these proofs heavily use properties of harmonic functions, and so far they are not extended to solutions of the Helmholtz equation. This extension is quite desirable since many inverse problems use prospecting by time harmonic waves.

While there are no global uniqueness results for general convex $D$ in the absence of interior sources, we can prove uniqueness of such $D$ when a field is generated by a source inside $D$.

We consider the conductivity equation

$$\text{div}(-a\nabla u) = f \quad \text{in } \mathbb{R}^3, \qquad \lim_{|x|\to\infty} u(x) = 0. \tag{4.3.5}$$

Let $D$ be a subdomain of a bounded domain $\Omega$, and $\Gamma$ a $C^1$-surface in $\mathbb{R}^3\backslash\overline{\Omega}$. We assume that $f$ and $k$ are given. For the inverse conductivity problem we consider the additional data

$$u = g_0, \quad \partial_\nu u = g_1 \quad \text{on } \Gamma. \tag{4.3.6}$$

**Theorem 4.3.7.** *Let us assume that* $f \geq 0$, $f \in L_1(\Omega)$; $f = 0$ *outside* $D$, $f \not\equiv 0$. *Let D be a convex domain and* $a = k\chi(D)$ *where*

$$k > 0 \quad \text{on } \Omega. \tag{4.3.7}$$

*Then D is uniquely determined by the data* (4.3.6).

In the proof we make use of the following results.

**Lemma 4.3.8.** *If two domains* $D_1$ *and* $D_2$ *produce the same data* (4.3.6), *then*

$$\int_{\partial D_1} v\partial_\nu u_1^i = \int_{\partial D_2} v\partial_\nu u_2^i$$

*for all functions* $v$ *harmonic near* $\overline{D_1} \cup \overline{D_2}$.

PROOF. From the definition (4.0.3) of the equality $\text{div}(a_1\nabla u_1) = \text{div}(a_2\nabla u_2)$ in $\mathbb{R}^3$ we get

$$\int_{\partial\Omega_0} \partial_\nu u_1 v - \int_{\Omega_0} a_1\nabla u_1 \cdot \nabla v = \int_{\partial\Omega_0} \partial_\nu u_2 v - \int_{\Omega_0} a_2\nabla u_2 \cdot \nabla v$$

for any function $v \in C^1(\overline{\Omega}_0)$, where $\Omega_0$ is a smooth domain containing $\overline{D_1} \cup \overline{D_2}$. As above, the equality (4.3.6) implies that $u_1 = u_2$ on $\Omega\backslash(\overline{D_1} \cup \overline{D_2})$. Let us choose $\Omega_0$ to be contained a neighborhood of $\overline{D_1} \cup \overline{D_2}$ where $v$ is harmonic. Then the boundary integrals are equal, and using (4.3.2), we conclude that

$$\int_{\Omega_0} \nabla u_1 \nabla v + k\int_{D_1} \nabla u_1 \cdot \nabla v = \int_{\Omega_0} \nabla u_2 \cdot \nabla v + k\int_{D_2} \nabla u_2 \cdot \nabla v.$$

Integrating by parts in the integrals over $\Omega$ and using the harmonicity of $v$ and the equality $u_1 = u_2$ outside $D_1 \cup D_2$, we conclude that these integrals are equal.

Then similarly, integrating by parts in the integrals over $D_1$, $D_2$, we arrive at the conclusion of this lemma. □

**Lemma 4.3.9.** *Under the conditions of Theorem* 4.3.7 *we have* $\partial_\nu u^i < 0$ *on* $\partial D$.

PROOF. Let

$$h(x) = (1+k)\int_D f(y)G(x,y)\,dy,$$

where $G(x,y) = 1/(4\pi|x-y|)$ is the standard fundamental solution of the Laplace operator. We have $\operatorname{div}(-a\nabla h) = f$ in $D$, and $h$ is harmonic outside $\overline{D}$. Since $D$ is convex, we have

$$\partial_{\nu(x)}G(x,y) = -(4\pi|x-y|^3)^{-1}\nu(x)\cdot(x-y) \le 0$$

when $x \in \partial D$ and $y \in D$, so $\partial_\nu h < 0$ on $\partial D$. When $u$ solves the direct problem, then the function $w = u - h$ is harmonic in $D$, and outside $\overline{D}$, it goes to zero at infinity, is continuous in $\mathbb{R}^3$, and satisfies the refraction condition

$$\partial_\nu w^e - (1+k)\partial_\nu w^i = k\partial_\nu h \quad \text{on } \partial D. \tag{4.3.8}$$

Moreover, any such $w$ generates a solution to the original problem for $u$. To find $w$ one can make use of the classical single layer potential

$$w(x) = U_1(x;\rho) = \int_{\partial D} \rho(y)G(x,y)d\Gamma(y),$$

with density $\rho$ to be found. The well-known jump relations for first order derivatives of this potential $\partial_\nu w^e = -\rho/2 + \partial_\nu U_1$, $\partial_\nu w^i = \rho/2 + \partial_\nu U_1$ on $\partial D$ (e.g., [Is4], section 1.6, [Mi], p. 35, for smooth $D$ and [EsFV] for Lipschitz $D$) where $\partial_\nu U_1$ denotes the potential on $\partial D$ with the kernel $\partial_\nu G$ transform the refraction condition (4.3.8) into the integral equation

$$-\rho/2 - (1+k)\rho/2 - k\partial_\nu U_1 = k\partial_\nu h \quad \text{on } \partial D$$

or

$$(I - K)\rho = -k/(1+k/2)\partial_\nu h \tag{4.3.9}$$

where $I$ is the identity operator and $K\rho = -k/(1+k/2)\partial_\nu U_1$.

The right side of the equation (4.3.9) is positive on $\partial D$. Since the kernel $\partial_\nu G$ of the operator $\partial_\nu U_1$ is non positive the operator $K$ maps nonnegative functions into nonnegative functions. According to the results of Escauriaza, Fabes, and Verchota the eigenvalues of the operator $\rho \to \partial_\nu U_1$ are contained in $(-1/2, 1/2]$. Due to the convexity assumptions, this operator has non-positive kernel, so its spectral radius in $L_2$ does not exceed $1/2$, the spectral radius of $K$ in $L_2$ is less than 1, and a solution $\rho$ to equation (4.3.9) is the sum of the Neumann series $(I + K + K^2 + \ldots)h_1$, where $h_1$ is the right side of (4.3.9). Since $h_1$ is positive, so is $\rho$.

The proof will be complete when we write

$$\partial_\nu u^e = \partial_\nu h + \partial_\nu w^e = \partial_\nu h - \rho/2 + \partial_\nu U_1$$

and express $\partial_\nu U_1$ from (4.3.9) to conclude that $\partial_\nu u^e = -(1/k+1)\rho < 0$. □

PROOF OF THEOREM 4.3.7. We will modify the orthogonality method used to prove Theorem 4.1.1.

Assume that there are two different domains $D_1$, $D_2$ producing the same exterior data. If one of them is contained in another, then we obtain a contradiction by using the argument from the beginning of Section 4.3. So we can assume that both $D_1 \backslash D_2$ and $D_2 \backslash D_1$ are nonempty. We denote by $\Gamma_{1e}, \Gamma_{1i}, \Gamma_{2e}, \Gamma_{2i}$ the exterior and interior parts $\partial D_1 \backslash \overline{D_2}$, $\partial D_2 \cap D_1$, $\partial D_2 \backslash \overline{D_1}$, $\partial D_1 \cap D_2$ of their boundaries. By using harmonic approximation and stability of the Dirichlet problem with respect to a domain (see, e.g., [Is4]) we can extend the orthogonality relation of Lemma 4.3.8 onto functions $v$ harmonic in $D_1 \cup D_2$ and continuous on the closure of this star-shaped domain. Let $g_0$ be any continuous function on $\partial(D_1 \cup D_2)$, $0 \le g_0 \le 1$, and $v$ a solution of the Dirichlet problem in $D_1 \cup D_2$ with the data $g_0$. We choose $g_0 = 0$ on some open part of $\Gamma_{2e}$. By using the comparison principle one can show then that $v < 1 - \varepsilon(F)$ for any compact set $F$ in $D_1 \cup D_2$ (see, e.g., [Is4], Lemma 1.7.4). By maximum principles $0 \le v$.

From Lemma 4.3.8 we have

$$\int_{\Gamma_{1e}} g_0 \partial_\nu u_1^i + \int_{\Gamma_{2i}} v \partial_\nu u_1^i = \int_{\Gamma_{2e}} g_0 \partial_\nu u_2^i + \int_{\Gamma_{1i}} v \partial_\nu u_2^i.$$

By using the inequalities for $v$ and Lemma 3.4.8 we bound the left side from above and the right side from below to obtain

$$\int_{\Gamma_{1e}} g_0 \partial_\nu u_1^i \ge \int_{\Gamma_{2e}} g_0 \partial_\nu u_2^i + \int_{\Gamma_{1i}} \partial_\nu u_2^i + \varepsilon,$$

where $\varepsilon$ does not depend on $g_0$. We can approximate in $L_1(\Gamma_{2e} \cup \Gamma_{1e})$ by such $g_0$ the function that is 1 on $\Gamma_{1e}$ and 0 on $\Gamma_{2e}$, which give the inequality

$$\int_{\Gamma_{1e}} \partial_\nu u_1^i - \int_{\Gamma_{1i}} \partial_\nu u_2^i > 0.$$

Observe that according to the refraction conditions (4.3.3) we have

$$\partial_\nu u_1^i = (1+k)^{-1} \partial_\nu u_1^e = (1+k)^{-1} \partial_\nu u_2^e \text{ on } \Gamma_{1e}$$

because $u_1^e = u_2^e$ outside $D_1 \cup D_2$, and similarly, $\partial_\nu u_2^i = (1+k)^{-1} \partial_\nu u_2^e$ on $\Gamma_{1i}$, so we can replace $u_1^i$ and $u_2^i$ by $u_2^e$ in the above integral. Since $u_2^e$ is harmonic in $D_1 \backslash \overline{D_2}$ and $\nu$ is the interior normal on $\Gamma_{1i}$, the integral must be zero.

We have a contradiction, which shows that the initial assumption $D_1 \ne D_2$ was wrong.

The proof is complete. □

The orthogonality relations can also be used to estimate some functionals of the conductivity coefficient $a$. Indeed, from the definition of a weak solution (4.0.3) we have

$$\int_\Omega a\nabla u \cdot \nabla\phi = \int_{\partial\Omega} g_1\phi,$$

and the right side is known from the data (4.0.6). In particular, when $a$ has the form (4.3.2),

$$\int_\Omega \nabla u \cdot \nabla\phi + k\int_D \nabla u \cdot \nabla\phi = \int_{\partial\Omega} g_1\phi \quad \int_\Omega \nabla u_0 \cdot \nabla\phi = \int_{\partial\Omega} \partial_\nu u_0\phi, \tag{4.3.10}$$

if $u_0$ is a harmonic function.

**Exercise 4.3.10.** Prove that the data (4.0.2), (4.0.6) of the inverse conductivity problem with $a$ of the form (4.3.2) in a unique and stable way determine the integrals

$$k\int_{\partial D} u\partial_\nu v$$

for any harmonic function $v$ in $\Omega$.

Now we will show how to use the orthogonality relations (4.3.10) to evaluate size of $D$. The most recent and advanced paper on this subject is written of Alessandrini, Rosset, and Seo [AlRS] where one can find more general and detailed results as well as references to previous work. The idea is to compare solutions of the problems with $a = 1 + k\chi(D)$ and with $a = 1$. We will assume that $u = u_0 = g_0$ on $\partial\Omega$. Choosing in (4.3.10) the test function $\phi = u - u_0$, subtracting the integral equalities (4.3.10) and utilizing that $\phi = 0$ on $\partial\Omega$ we yield

$$\int_\Omega |\nabla u - \nabla u_0|^2 + k\int_D \nabla u \cdot \nabla(u - u_0) = 0.$$

From the first equality (4.3.10) with $\phi = u_0$,

$$k\int_D \nabla u \cdot \nabla u_0 = \int_{\partial\Omega} \partial_\nu u u - \int_\Omega \nabla u \cdot \nabla u_0 = \int_{\partial\Omega} (\partial_\nu u - \partial_\nu u_0)g_0$$

where we also used the second equality (4.3.10) with $\phi = u_0$. With the aid of the last integral identity we finally arive at the relation

$$\int_\Omega |\nabla u - \nabla u_0|^2 + k\int_D \nabla u \cdot \nabla u = \int_{\partial\Omega} (\partial_\nu u - \partial_\nu u_0)g_0. \tag{4.3.11}$$

If $0 < k$ then the both terms on the left side are nonnegative and therefore bounded by the known integral over $\partial\Omega$. By using the elementary inequality $b^2 \leq 2(|b - c|^2 + c^2)$ we can eliminate unknown term with $u$ in the integral (4.3.11) over $\Omega$:

$$k\int_D |\nabla u_0|^2 \leq 2k(\int_D |\nabla(u - u_0)|^2 + \int_D |\nabla u|^2) \leq 2(k+1)\int_{\partial\Omega} (\partial_\nu u - \partial_\nu u_0)g_0.$$

This inequality can be used in the estimation of size of $D$. For example, letting $u_0(x) = x_j$ we will have

$$k/2(k+1) vol D \leq \int_{\partial\Omega} (\partial_\nu u(;j) - \nu_j) x_j$$

where $u(;j)$ solves the Dirichlet problem with the data $u(x;j) = x_j$, $x \in \partial\Omega$.

Similarly one obtains the integral identity

$$\int_\Omega (1 + k\chi(D))|\nabla(u - u_0)|^2 - k \int_D |\nabla u_0|^2 = \int_{\partial\Omega} (\partial_\nu u_0 - \partial_\nu u) g_0,$$

which can be used to evaluate $vol D$ when $-1 < k < 0$ and also for bounding the right side of (4.3.11) by the integral over $D$ to conclude that

$$(4.3.12) \quad 1/k \int_\Gamma (\partial_\nu u - \partial_\nu u_0) g_0 \leq \int_D |\nabla u_0|^2 \leq 2(1 + 1/k) \int_\Gamma (\partial_\nu u - \partial_\nu u_0) g_0$$

In [AlRS] they considered more general elliptic equations and by using so-called doubling inequalitites to bound the inner integral in (4.3.12) by $vol D$ obtained two-sided estimates of $vol D$ for arbitrary nonzero Neumann boundary data with constants depending on this data. A good review of available results on size estimation (including evaluation of elastic cavities) is given in the paper of Alessandrini, Morassi, and Rosset in [I3].

We observe that for nonlinear conductivity equation (4.3.1) with $a = a(u)$ there is a global uniqueness and certain existence results due to Pilant and Rundell [PiR3]. We demonstrate their method in a slightly more general many-dimensional situation. We consider a domain $\Omega \subset \mathbb{R}^n$ with the $C^{2+\lambda}$-boundary. Let $\Gamma$ be a $C^1$-curve on $\partial\Omega$. We prescribe the following boundary value data:

$$a(u)\partial_\nu u = g_1 \quad \text{on } \partial\Omega, \qquad g_1 \in C^{1+\lambda}, \int_{\partial\Omega} g_1 = 0$$

and the additional boundary data for determination of $a$,

$$u = g_0 \text{ on } \Gamma.$$

Let $b$ be a point of $\partial\Omega$ and

$$v(x) = \int_{u(b)}^{u(x)} a(s)ds.$$

Then equation (4.3.1) and the Neumann boundary conditions (4.0.6) are transformed into the following equation and boundary conditions:

$$(4.3.13) \quad \begin{aligned} -\Delta v &= 0 \text{ in } \Omega, \\ \partial_\nu v &= g_1 \text{ on } \partial\Omega; \end{aligned}$$

and the additional boundary data give

$$(4.3.14) \quad \int_{g_0(b)}^{g_0(x)} a = v(x) \quad \text{when } x \in \Gamma.$$

Since $v(b) = 0$, equations (4.3.10) uniquely determine $v$, and then one can determine $a$ on the range of $g_0$ from the last equation.

One can prescribe the Dirichlet data $g_0 \in C^{2+\lambda}$ in the direct problem and the additional Neumann data on $\Gamma$ that are transformed into the boundary relations (4.3.14) on $\partial\Omega$ and (4.3.13) on $\Gamma$. Uniqueness of $a$ on the range of $g_0$ when the maximum and minimum of $g_0$ are achieved on $\Gamma$ still holds, as shown below.

Let $a_1, a_2$ be two conductivity coefficients and $v_1, v_2$ the corresponding functions $v$. Let $a = a_2 - a_1$, $w = v_2 - v_1$. Then subtracting the relations (4.3.13), (4.3.14), we obtain

$$\Delta w = 0 \text{ in } \Omega,$$
$$\partial_\nu w = 0 \text{ on } \Gamma, \qquad \int_{g_0(b)}^{g_0(x)} a = w(x) \quad \text{on } \partial\Omega.$$

By Giraud's extremum principle, a nonconstant $w$ cannot attain its maximum or minimum over $\overline{\Omega}$ at a point of $\Omega \cup \Gamma$, so they are attained on $\partial\Omega\backslash\Gamma$. Since $a$ is positive, this means that the maximum and minimum of $g_0$ are on $\partial\Omega\backslash\Gamma$ as well, which contradicts the choice of $g_0$. Therefore, $w$ is constant on $\Omega$, which must be zero because $w(b) = 0$. Then $a_1 = a_2$ on the range of $g_0$.

The choice of the Dirichlet data for the direct problem has advantages, because then the range of $g_0$ is under control.

## 4.4 Methods of the theory of one complex variable

In the plane case one can enjoy additional opportunities of the theory of functions of one complex variable. In particular, conformal mappings and Riemann-Hilbert type boundary value problems are very useful when studying the free boundary problems where a domain $D$ is to be found from certain exterior data.

We introduce the complex variables $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$ and the related complex differentiations

(4.4.1) $$\partial = 2^{-1}(\partial_1 - i\partial_2), \quad \bar\partial = 2^{-1}(\partial_1 + i\partial_2).$$

The function $\bar\partial u$ then is one-half of the gradient of $u$. It is easy to show that $\Delta = 4\partial\bar\partial$. For references on theory of one complex variable we refer to the book of Ahlfors [Ah].

**Lemma 4.4.1.** *Let $D$ be a bounded simply connected domain in $\mathbb{R}^2$ with rectifiable boundary. Let $f \in L^\infty(D)$. Let $u$ be a function that is harmonic in $\mathbb{R}^2\backslash\overline{D}$, is $C \ln|x| + u_0$, where $u_0$ goes to zero at infinity and $u_0 \in C^1(\mathbb{R}^2\backslash D)$. Let $U(;f\chi(D))$ be the logarithmic potential of $D$ with density $f$.*

*Then the equality*

(4.4.2) $$U(;f\chi(D)) = u \quad \textit{outside } D$$

*is equivalent to the equality*

$$\partial u = F + \phi^+ \quad on\ \partial D, \tag{4.4.3}$$

*where F is a solution to the equation* $-4\bar\partial F = f$ *in* $D$, $F \in C(\overline{D})$, *and* $\phi^+ \in C(\overline{D})$ *is a complex analytic function on* $D$.

PROOF. The potential $U(; f\chi(D))$ is contained in $C^1(\mathbb{R}^2)$, and $-4\partial\bar\partial U(; f\chi(D)) = f\chi(D)$. Since $u$ and the potential are equal outside $D$, their antigradients $\partial$ are equal on $\partial D$. Let $\phi^+ = \partial U(; f\chi(D)) - F$. Then $\bar\partial\phi^+ = 0$ on $D$, and we have the equality (4.4.3).

Now let us start with (4.4.3). Define the function $V$ as $\partial U(; f\chi(D)) - \partial u$ on $\mathbb{R}^2\backslash D$ and as $\partial U(; f\chi D)) - F - \phi^+$ in $D$. Both functions outside and inside of $D$ are analytic functions, continuous up to $\partial\overline{D}$ and equal on $\partial D$. By the continuity principle, $V$ is complex analytic in $\mathbb{C}$. In addition, gradients of $u$ and of the potential go to zero at infinity, so $V$ is bounded. By Liouville's theorem, it is constant, which must be zero due to the behavior at infinity. Since the gradient of the difference of $u$ and the potential is zero, this difference is constant outside $D$, which turns out to be zero if we remember the behavior at infinity.

The proof is complete. □

Let $z(t)$ be a conformal mapping of the unit disk $B(0; 1)$ onto $D$. Since $D$ is Jordan, $z(t)$ is continuous on the closure of the unit disk. Letting $U = \partial u$ and transplanting the relation (4.4.3) onto the boundary of this disk, we obtain

$$U(z(t)) = F(z(t), \overline{z(t)}) + \phi^+(t) \quad \text{when } |t| = 1,$$

where $\phi^+$ is a function that is complex analytic in the unit disk and continuous on its closure. In particular, when $f$ is constant, we may choose $F(z, \bar z) = -f\bar z/4$, which gives the useful relation

$$U(z(t)) = -f\bar z(1/t)/4 + \phi^+(t) \quad \text{when } t\bar t = |t|^2 = 1, \tag{4.4.4}$$

which helps, say, to analyze the connection between regular points of the exterior potential $\partial u$ and the conformal map $z(t)$. The relation (4.4.4) has been obtained by V. Ivanov [Iv].

Indeed, let $D_\bullet$ be a domain containing $\mathbb{C}\backslash\overline{D}$, $\Gamma$ the part of $\partial D$ inside $D_\bullet$, and $\gamma$, $G_\bullet$ be the inverse images of $\Gamma$, $D \cap D_\bullet$ under the mapping $z(t)$. The exterior potential $U(; f\chi(D))$ has a harmonic continuation from $\mathbb{C}\backslash\overline{D}$ onto $D_\bullet$ if and only if $z(t)$ has an analytic continuation onto the interior of $B(0; 1) \cup \gamma \cup G^*_\bullet$, where $G^*_\bullet$ is the image of $G_\bullet$ under the inversion $t \to 1/\bar t$.

If the exterior potential $u$ has a continuation, then the left side of (4.4.4) is complex analytic in $G_\bullet$. $\bar z(1/t)$ is complex analytic outside of the unit disk and continuous up to its boundary. Since it is equal on $\gamma$ to the function $4/f(\phi^+(t) - U(z(t))$, which is complex analytic on $G_\bullet$ and continuous up to $\gamma$, by the continuity principle we conclude that $\bar z(1/t)$ has an analytic continuation onto $G_\bullet$. Applying inversions, we obtain the claim for $z(t)$.

If the conformal mapping has a continuation, then we can use the same argument to show that $U$ has an analytic continuation onto $D_\bullet$, and therefore $u$ has a harmonic continuation.

The relation (4.4.4) can be considered as a nonlinear boundary value problem for analytic functions $z(t)$, $\phi^+(t)$ in the unit disk. This problem can be solved at least locally, which gives also some constructive method of finding $D$ given the exterior potential $u$.

We refer for the corresponding results and details to the book of Cherednichenko [Cher] and of Isakov [Is4] and the references given there.

A similar technique is even more useful when studying the inverse conductivity problem.

Let $D$ be a simply connected subdomain of $\Omega$ with the Lipschitz boundary. Let $u$ be a solution to the conductivity problem (4.3.3) with the Dirichlet data $u = g$ on $\partial\Omega$. By using the refraction condition on $\partial D$ it is easy to observe that the integral of $\partial_\nu u^e$ over $\partial D$ is $(1+k)$ times the integral of $\partial_\nu u^i$, which is zero because the function $u^i$ is harmonic inside $D$. Therefore, we can find the harmonic conjugate $v^e$ of $u^e$ in $\Omega\backslash\overline{D}$. Since $D$ is simply connected, there is the harmonic conjugate $v^i$ to $u^i$ in $D$. We can assume that both $v^e$ and $v^i$ are equal to zero at some point of $\partial D$. We assume that the pair $(\tau, \nu)$, where $\tau$ is a unit tangent to $\partial D$, is oriented as the coordinate vectors $e_1$, $e_2$, and we recall the following form of the Cauchy-Riemann equation for $u, v$:

$$\partial_\tau u = \partial_\nu v, \qquad \partial_\tau v = -\partial_\nu u, \tag{4.4.5}$$

where $u$ should be replaced by $u^e$ outside $D$ and by $u^i$ inside $D$ and so is $v$. We introduce the complex analytic functions $U^e = u^e + iv^e$ and $U^i = u^i + iv^i$. Since we are given the Cauchy data for $u^e$ on $\Gamma$, this function can be considered as given.

Let $z(t)$ be the conformal mapping of the unit disk onto $D$ normalized in the standard way: $z(0) = 0$, $z'(0) > 0$.

**Lemma 4.4.2.** *A domain $D$ is a solution to the inverse conductivity problem with the exterior data $u^e$ if and only if*

$$\psi^+(t) = (2+k)U^e(z(t)) + k\overline{U}^e(z(t)) \text{ when } |t| = 1 \tag{4.4.6}$$

*for some function $\psi^+$ that is complex analytic in $B(0;1)$ and whose first-order derivatives are in $L_2(\partial B(0;1))$.*

PROOF. Let $D$ be a solution to the inverse conductivity problem. From the Cauchy-Riemann conditions (4.4.5) and from the refraction conditions (4.3.3) we conclude that $\partial_\tau v^e = (1+k)\partial_\tau v^i$ on $\partial D$. Integrating along $\partial D$ and using the special choice of $v^e, v^i$, we conclude that $v^e = (1+k)v^i$ on $\partial D$. This relation and the continuity of the function $u$ yield

$$U^e + \overline{U}^e = U^i + \overline{U}^i, \quad (U^e - \overline{U}^e) = (1+k)(U^i - \overline{U}^i) \quad \text{on } \partial D.$$

By substituting $\overline{U}^i$ from the first relation into the second one we obtain the equality (4.4.6) with $\psi^+(t) = (2+2k)U^i(z(t))$.

Now let the relation (4.4.6) be valid, where $z(t)$ is the normalized conformal map of the unit disk onto $D$. Let $U_D^e$ be the exterior analytic function constructed for the exterior part of the solution of the direct conductivity problem. Then we have the relation (4.4.6) for $U_D$ with $\psi^+$ replaced by $(2+2k)U_D^i(z(t))$. Let $U^i(z) = \psi^+(t(z))/(2+2k)$. Subtracting the relations (4.4.6) for $U$ and $U_D$ and letting $U_\bullet$ be their difference, we obtain the equality $(2+2k)U_\bullet^i = (2+k)U_\bullet^e + k\overline{U}_\bullet^e$ on $\partial D$. Since $\Re U^e$ and $\Re U_D^e$ have the same Dirichlet data on $\partial\Omega$, we conclude that $\Re U_\bullet^e = 0$ on $\partial\Omega$. Subtracting the boundary relation for $U_\bullet$ and its complex adjoint as well as adding these relations, we arrive at the equalities

$$U_\bullet^e - \overline{U}_\bullet^e = (1+k)(U_\bullet^i - \overline{U}_\bullet^i), \quad U_\bullet^e + \overline{U}_\bullet^e = U_\bullet^i + \overline{U}_\bullet^i,$$

or

$$u_\bullet^e = u_\bullet^i, v_\bullet^e = (1+k)v_\bullet^e \quad \text{on } \partial D.$$

Differentiating the second relation in the tangential direction and using once more the Cauchy-Riemann conditions (4.4.5), we conclude that $\partial_\nu u_\bullet^e = (1+k)\partial_\nu u_\bullet^i$ on $\partial D$. So $u_\bullet$ is a solution to the direct conductivity problem with zero Dirichlet data on $\partial\Omega$, and it follows that $u_\bullet^e = 0$ outside $D$. Therefore, $u^e = u_D^e$, and the proof is complete. □

The relation (4.4.6) can be considered as a nonlinear boundary value problem for analytic functions $\psi^+$, $z$ in the unit disk whose solution (with given $U^e$) produces a solution $D$ to the inverse conductivity problem. In particular, one can obtain analyticity results similar to those for inverse gravimetry, provided that $\partial U^e \neq 0$ on $\partial D$ (or $\nabla u^e \neq 0$, which is the same). This condition is controlled by the index (winding number) of the vector field $\nabla u^e$, and it plays a crucial role in Theorem 4.4.3.

In proving analyticity and even uniqueness results similar to Theorem 4.3.5 one can use the Schwarz function and complete regularity results for domains having such a function that were obtained by Sakai [Sa]. Let $\gamma$ be a portion of $\partial D$ for an open set $D \subset \mathbb{C}$. Let the origin be a point of $\Gamma$. A function $S$ defined on $D \cup \Gamma$ is called a Schwarz function of $D \cup \Gamma$ if $S$ is complex analytic in $D$, continuous in $D \cup \Gamma$, and $S(z) = \overline{z}$ on $\Gamma$. Sakai showed that loosely speaking, $\Gamma$ near the origin is either a regular analytic curve or the union of two such curves or a regular analytic cusp directed into $D$. If the exterior potential $U$ has a complex analytic continuation across $\Gamma$, then letting $t = t(z)$ in (4.4.4), we obtain the relation $\overline{z} = 4(\phi^+(z) - U(z))$ on $\Gamma$, so the right side is a Schwarz function, and we obtain analyticity of $\Gamma$. Similarly, if we can resolve equation (4.4.6) with respect to $z$, we obtain the same claim. This equation can be locally resolved when $\partial U \neq 0$ on $\Gamma$, which can be guaranteed by index theory, as we will show later.

We will assume that $D = D_\sigma$ is close to a $C^{1+\lambda}$-domain $D_0$ in the following sense. Let $\Sigma$ be a family of complex analytic functions $\sigma$ on $B(0;1)$, $|\sigma|_{1+\lambda}(B(0;1)) \leq M$, such that $z(t) + \sigma(t)$ is the normalized conformal mapping of $B(0;1)$ onto some simply connected domain $D_\sigma$. A sufficient condition is that $\sigma(0) = 0$, $\Im\sigma'(0) = 0$, and $M$ is small. We will assume that $\Sigma$ is closed in $C^{1+\lambda}(B(0;1))$.

**Theorem 4.4.3.** *Let $\partial D_0$ be $C^{1+\lambda}$. Let us assume that the boundary data $g_0 \in C^2(\partial\Omega)$ have a unique local maximum and unique local minimum on $\partial\Omega$.*

*For the family $\Sigma$ there is a number $\varepsilon_\Sigma$ such that if $D_\sigma$ is a solution to the inverse conductivity problem and $|\sigma|_0(B(0;1)) < \varepsilon$, then $D_\sigma = D_0$.*

A complete proof of this result is given in the papers of Alessandrini, Isakov, and Powell [AlIP] and of Powell [Pow]. It is based on linearization of the boundary condition (4.4.6), the study of the corresponding linear problem $A(\psi, \sigma) = \psi_1$, and the contracting remainder $B$. The linearization is represented by the boundary value problem for complex analytic functions $\psi, \sigma$ in the unit disk $B(0;1)$ with the nonlinear boundary condition

$$A(\psi, \sigma)(t) = B\sigma(t) \qquad \text{when } |t| = 1, \tag{4.4.7}$$

where

$$A(\psi, \sigma) = \psi - (2 + k)a\sigma - k\overline{a\sigma},\ a(t) = \partial u(z(t))$$

and

$$B\sigma = (2 + k)B_1\sigma + \overline{B_1\sigma},\ B_1\sigma = U^e(z + \sigma) - U^e(z) - \partial u^e(z)\sigma.$$

Here $A$ is considered as an operator from $\Psi \times \Sigma$ into $C^\lambda(\partial B(0;1))$. We define $\Psi$ as the space of functions complex analytic in $B(0;1)$ and Hölder in its closure, and $\Sigma$ as the space of functions $\sigma \in \Psi$ such that $\sigma(0) = 0$, $\Im\sigma'(0) = 0$. The operator $A$ is continuous from $\Psi \times \Sigma$ onto its range $\Re \subset C^\lambda(B(0;1))$, which is known to be finite-codimensional. The main claim about $A$ is that under the conditions of Theorem 4.4.3 its kernel is zero, and therefore this Fredholm operator is invertible from $\Re$ into $\Psi \times \Sigma$. A proof of this claim given in [AlIP] is based on Muskhelishvili's [Mus] theory of one-dimensional singular integral equations (as suggested and proven in the paper of Bellout, Friedman, and Isakov [BelFI] for analytic $\partial D_0$). The crucial step is to show that the index (winding number) of the vector field $a$ is well-defined on $\partial B(0;1)$ and is zero. We observe that the boundary value problem for the conformal map was derived and used by Cherednichenko [Cher].

First, we observe that a solution to the elliptic differential equation (4.3.1) in $\mathbb{R}^2$ cannot have a zero of infinite order unless this solution is constant (see Section 3.3). This observation implies that all zeros of the gradient of a nonconstant solution $u$ to the equation $\operatorname{div}((1 + k\chi(D))\nabla u) = 0$ in $\Omega$ are isolated, and $u$ near a zero admits the following representation: $u(x) - u(x_0) = H_N(x - x_0) + O(|x - x_0|^{N+\lambda})$, where $H_N$ is a homogeneous function of degree $N$ ([AlIP], Theorem 4). the geometric interpretations of the index of $\nabla u$ around zero gradient given in the papers of Alessandrini and Magnanini [AlM] and of Alessandrini, Isakov, and Powell [AlIP] as the multiplicity $N$ of a zero of the gradient and homotopic invariance of the index show that the sum of indices of all critical points of $u$ inside

$\Omega$ is equal to the index of $\nabla u$ over $\partial\Omega$. From the conditions of Theorem 4.4.3 on the boundary data we obtain that the index of $\nabla u$ over $\partial\Omega$ is 0. So there is no zero of $\nabla u^e$ in $\Omega\backslash\overline{D}_0$ and of $\nabla u^i$ in $\overline{D}$. Hence the index of $a = \partial u$ over $\partial D$ is zero. Another way to understand why the conditions on $g_0$ imply the absence of zeros of $\nabla u$ inside $\Omega$ is to use level curves of $u$. If there is a zero inside, then more than two level curves are entering $\partial\Omega$, which contradicts the condition on $g$.

It is interesting that the linearized problem for analytic functions can be transformed into the oblique derivative problem for harmonic functions $v$ in $D_0$ with the boundary condition $\nabla u^e \cdot \nabla v = 0$ on $\partial D_0$, as observed by Bellout, Friedman, Isakov [BelFI]. Our conditions guarantee that this problem is elliptic in the two-dimensional case, but it is always nonelliptic in higher dimensions. This is a partial explanation why even local uniqueness results are not available in the three-dimensional case. There are expectations of obtaining local uniqueness results from three special boundary measurements.

One can anticipate conditional logarithmic (local) stability in the situation of Theorem 4.4.3 which probably can be obtained by using sharp stability results for analytic/harmonic continuation up to the boundary of $D$ (like in [AlBRV], [Ron] for different problems) and standard elliptic theory which implies that the inverse of the linearized operator $A$ in (4.4.7) is continuos from $C^{1+\lambda}$ into itself. Logarithmic stability should be optimal as suggested by the general theory in Di Christo, Rondi [DR].

**Exercise 4.4.4.** Prove that if $g_0$ is chosen as in Theorem 4.4.3, then Theorem 4.3.5 about convex polygons is valid without the assumption (4.3.4).

{*Hint*: Use that the index of $\nabla u$ over $\partial\Omega$ is zero and that from the proof of Lemma 4.3.6 one can conclude that if $u^e$ has a harmonic continuation onto a neighborhood of a vertex, then this vertex must be a zero of $\nabla u^e$.}

## 4.5 Linearization of the coefficients problem

The problems of identification of coefficients are quite complicated, in particular due to their nonlinearity. In practical computations one often linearizes these inverse problems, and the results of reconstruction via linearization are sometimes very useful. The linearization of the coefficients problem is an inverse source problem. In this section we justify linearization in two important cases: for smooth and for special discontinuous perturbations. We discuss the linearized problems in this section and later.

Let us consider the Dirichlet problem

$$\begin{aligned} &\operatorname{div}(-a\nabla u) + b\cdot\nabla u + cu = f \text{ in } \Omega,\\ &u = g_0 \text{ on } \partial\Omega. \end{aligned} \tag{4.5.1}$$

We assume that $a = a_0 + a_\delta, \ldots, c = c_0 + c_\delta$ are some perturbations of the coefficients of the equation such that it remains uniformly elliptic under these perturbations. Let $u_0$ be a solution to (4.5.1) with $a_\delta = 0, \ldots, c_\delta = 0$, $u_\delta$ a solution to (4.5.1), and $v_\delta$ equal to $u_\delta - u_0$. We are interested in comparing $v_\delta$ with a solution $v$ to the problem

$$\begin{aligned} \operatorname{div}(-a_0\nabla v) + b_0 \cdot \nabla v + c_0 v &= \operatorname{div}(a_\delta \nabla u_0) - b_\delta \cdot \nabla u_0 - c_\delta u_0 \text{ in } \Omega, \\ v &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.5.2}$$

**Lemma 4.5.1.** *Let*

$$\|a_\delta\|_\infty(\Omega) + \|b_\delta\|_\infty(\Omega) + \|c_\delta\|_\infty(\Omega) = \delta.$$

*Then* $\|v_\delta\|_{(1)}(\Omega) \le C\delta\|g_0\|_{(1/2)}(\partial\Omega)$ *and* $\|v - v_\delta\|_{(1)}(\Omega) \le C\delta^2\|g\|_{(1/2)}(\partial\Omega)$.

*If in addition* $b_\delta = 0$, *and either* $c_\delta = 0$ *and* $|\nabla u_0| > \varepsilon_0$ *on* $\Omega$ *or* $a_\delta = 0$ *and* $|u_0| > \varepsilon_0$ *on* $\Omega$ *for some fixed g, then*

$$\|a_\delta\|_{(0)}(\Omega) + \|c_\delta\|_{(0)}(\Omega) \le C\|v_\delta\|_{(1)}(\Omega).$$

PROOF. By subtracting equations (4.5.1) for $u_\delta$ and $u_0$ we obtain

$$\begin{aligned} &\operatorname{div}(-a\nabla v_\delta) + b \cdot \nabla v_\delta + c v_\delta \\ &\quad = \operatorname{div}(a_\delta \nabla u_0) - b_\delta \cdot \nabla u_0 - c_\delta u_0 \text{ in } \Omega, \\ &u_\delta = 0 \text{ on } \partial\Omega. \end{aligned}$$

From Theorem 4.1 and our assumptions on $a_\delta$, $b_\delta$, and $c_\delta$ it follows that

$$\|a_\delta \nabla u_0\|_{(0)}(\Omega + \|b_\delta \nabla u_0\|_{(0)}(\Omega) + \|c_\delta u_0\|_{(0)}(\Omega) \le C\delta,$$

so the estimates of Theorem 4.1 imply the first estimate of this lemma. By subtracting the equation for $v_\delta$ and $v$ we get

$$\begin{aligned} &\operatorname{div}(-a_0\nabla(v_\delta - v)) + b_0 \cdot \nabla(v_\delta - v) + c_0(v_\delta - v) \\ &\quad = \operatorname{div}(a_\delta \nabla v_\delta) - b_\delta \cdot \nabla v_\delta - c_\delta v_\delta \text{ in } \Omega, \\ &v_\delta = 0 \text{ on } \partial\Omega. \end{aligned}$$

Now the appropriate norms ($\|a_\delta \nabla v_\delta\|_{(0)}(\Omega) + \cdots$) of the right sides are bounded by $C\delta^2$ which follows from the first estimate; and arguing as above, we obtain the second estimate.

Let $c_\delta = 0$. From the equation for $v_\delta$ it follows that $\|a_\delta \nabla u_0\|_{(0)}(\Omega) \le C\|v_\delta\|_{(1)}(\Omega)$. Due to our conditions on $u_0$ this implies the bound of $\|a_\delta\|_{(0)}(\Omega)$. The proof when $a_\delta = 0$ is similar.

The proof is complete. □

The results of Lemma 4.5.1 somehow justify the linearization (replacement of $v_\delta$ by $v$) in the inverse coefficient problem when perturbations are $\delta$-small in the uniform norm and perturbations are a priori bounded in, say, Lipschitz norm, because in this case the $L_1$-norm bounds the $L_\infty$-norm. The conditions of Lemma 5.4.1 on $u_0$ can be guaranteed by an index-type argument or by maximum principles for some choice of the Dirichlet data $g_0$ (say, $g_0 > 0$ on $\partial\Omega$). In fact, we neglect terms of magnitude $\delta^2$ when the data are of magnitude $\delta$. However, it is not a genuine justification, because while we can guarantee that for the data $\partial_\nu u$ of the inverse problem we have $\|\partial_\nu(v_\delta - v)\|_{(-1/2)}(\partial\Omega) \le C\delta^2$ we cannot tell that $\|\partial_\nu v\|_{(-1/2)} \ge \delta/C$. Moreover, the last bound is generally wrong, due to nonuniqueness and ill-posedness of the linearized inverse problem, when in addition to (4.5.2) we prescribe

$$a_0\partial_\nu v = g_1 \text{ on } \partial\Omega. \tag{4.5.3}$$

A complete justification requires much more delicate analysis, and we will not do it in this book.

**Exercise 4.5.2.** Consider the linearized inverse problem (4.5.2), (4.5.3) when $a_0 = 1, a_\delta = k\chi(D), b_0 = b_\delta = 0,\ c_0 = c_\delta = 0$. Assume that $u_0(x) = x_n$ and $k$ is constant and known. Prove the uniqueness of the unknown $x_n$-convex $D$.

{*Hint*: Make use of the methods of the proofs of Theorem 4.3.7.}

Now we consider more special perturbations that are not uniformly small and that are more natural in those applications one is looking for shapes of unknown inclusions of relatively small volume.

We assume that $b = c = 0$ and $a_\delta = k\chi(D)$, where $D$ is an open set and $k$ is a function that is piecewise $C^2$-smooth on $D$. The linearization will be considered under the assumptions

$$\begin{gathered}\text{area}(\partial D) \le C,\ \|k\|_{2,\infty}(\Omega) \le C,\\ \text{vol}(D) = \delta \text{ is small, and } \text{dist}(D, \partial\Omega) \ge 1/C.\end{gathered} \tag{4.5.4}$$

When $\delta$ is small, so is $\|a_\delta \nabla u_0\|_{(0)}(\Omega)$. From now on we assume that $a_0 \in C^2(\overline{\Omega})$, $\partial\Omega \in C^2$ and $g_0 \in C^2(\partial\Omega)$. Then the known elliptic theory (Theorem 4.1) and embedding theorems give that $u_0 \in C^1(\overline{\Omega})$.

**Exercise 4.5.3.** In the notation above prove that $\|v_\delta\|_{(1)}(\Omega) \le C\delta^{1/2}$.

Let $G(x, y)$ be Green's function of the Dirichlet problem for the Laplace operator in $\Omega$. As above, the equations for $v_\delta$ read

$$-\Delta v_\delta = \text{div}(k\chi(D)\nabla u_0) + \text{div}(k\chi(D)\nabla v_\delta) \text{ in } \Omega, \quad v_\delta = 0 \quad \text{on } \partial\Omega. \tag{4.5.5}$$

Green's representation and integration by parts give

$$\begin{aligned} v_\delta(x) &= \int_D k\nabla G(x, )\cdot(\nabla u_0 + \nabla v_\delta)\\ &= \int_{\partial D} k\partial_\nu G(x, )(u_0 + v_\delta)d\Gamma - \int_D \nabla k\cdot\nabla G(x, )(u_0 + v_\delta), \end{aligned} \tag{4.5.6}$$

provided that $x \in \Omega\backslash\overline{D}$. Considering equation (4.5.5) for $v_\delta$ as an elliptic equation with the discontinuous coefficient $1 + k\chi(D)$ and with the right side $\text{div}(k\chi(D)\nabla u_0)$, and using the inequality $\|k\nabla u_0\|_p(D) \le C\delta^{1/p}$ with $p > n$, we deduce from the results of Exercise 4.5.2 and from known interior

Schauder-type estimates for equations in the divergent form given in Theorem 4.1 that $\|v_\delta\|_\infty(D) \le C\delta^{1/p}$. When we choose the Dirichlet data $g_0 > 0$ on $\partial\Omega$, the maximum principle for harmonic functions gives $u_0 > \varepsilon > 0$ on $\Omega$, so the integrals on the right side of (4.5.6) involving $v_\delta$ are much smaller than the integrals involving $u_0$, provided that $x \in \partial\Omega$. This is a partial justification of the linearization that consists in replacing the nonlinear inverse problem (4.5.5) by the linear one

$$-\Delta v = \mathrm{div}(k\chi(D)\nabla u_0) \quad \text{in } \Omega, \quad v = 0 \text{ on } \partial\Omega,$$

with the additional Neumann data $\partial_\nu v = h$ on $\partial\Omega$. The corresponding linearized inverse problem is easier to analyse. In particular, when $k$ is a known constant and $u_0$ is a linear function one can show uniqueness of $D$ which is convex in the direction of $\nabla u_0$. However, the remainder $v_\delta$ is not $o(\delta)$. A more detailed analysis shows that the genuine linearization is more complicated. To explain difficulities we mention the recent result of Beretta, Francini, and Vogelius [BerFV].

Let $\sigma$ be an orientable hypersurface in $\mathbb{R}^n$ of class $C^3$ with $\overline{\sigma} \subset \Omega$ with a fixed $C^2$-smooth unit normal $\nu$. Let $a_0$ be a constant, $a_\delta = k\chi(D_\delta)$ where $D_\delta = \{y : dist(y, \sigma) < \delta\}$. Let $b_0 = b_\delta = 0, c_0 = c_\delta = 0$. Let $(\tau_1, \dots, \tau_{n-1}, \nu)$ be an orthonormal basis in $\mathbb{R}^n$, with $\tau_1, \dots, \tau_{n-1}$ tangent to $\sigma$. We define $A(;\sigma, a_0, k)$ as a symmetric matrix with eigenvectors $\tau_1, \dots, \tau_{n-1}, \nu$ and corresponding eigenvalues $1, \dots, 1, a_0/(a_0 + k)$. One can assume that $A \in C^2(\sigma)$. It was shown in [BerFV] that

$$u(x) - u_0(x) = 2\delta \int_\sigma kA(y;\sigma)\nabla u_0(y) \cdot \nabla_y G(x, y) d\sigma(y) + r(x;\delta)$$

where $|r(;\delta)| \le C\delta^{1+\kappa}\|g\|_{(1/2)}(\partial\Omega)$, $\kappa \in (0, 1)$ depends on $n$ and $C$ depends on $n, \Omega, a_0/(a_0 + k), \sigma$. Most likely the result of the linearization depends on approximating domains $D_\delta$, which are chosen to be "uniformly thin" in [BerFV].

While the methods of section 4.4 are quite sharp and powerful they are restricted to the plane case. In any dimension linearization at fixed domains can be obtained by using the theory of domain derivatives exposed, for example, in the book of Sokolowski and Zolesio [SoZ].

## 4.6 Some problems of detection of defects

Locating defects inside of material, in particular, locating cracks, is a fundamental importance in (airplane and automotive) engineering. The mathematical study of this problem originated in the paper of Friedman and Vogelius [FV]. We will formulate some recent results.

Let $u_\sigma$ be a solution to the following boundary value problem:

$$(\Delta + k^2)u = 0 \text{ in } \Omega\backslash\sigma, \qquad \partial_\nu u = g_1 \text{ on } \partial\Omega;$$

$$u = C \text{ on } \sigma \text{ (insulating crack) or } \partial_\nu u = 0 \text{ on } \sigma \text{ (conducting crack).} \tag{4.6.1}$$

Here $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$; $C$ is a (unknown) constant at each connected component of $\sigma$, and $\sigma$ is a Lipschitz curve in $\Omega$ with connected $\Omega\backslash\sigma$ in $\mathbb{R}^2$ or a Lipschitz surface in $\mathbb{R}^3$. When $k = 0$, by minimizing the Dirichlet integral or by using conformal mapping of $\Omega\backslash\sigma$ onto an annulus, one can secure the existence and uniqueness of $u \in H_{(1)}(\Omega\backslash\sigma)$, provided that natural orthogonality relations are satisfied (the integrals of $g_1$, $u$ over $\partial\Omega$ are zero). By standard elliptic theory this solvability result will hold for all real $k$ except of some set accumulating to infinity.

First, we consider *buried* cracks; i.e., $\bar{\sigma} \subset \Omega$.

When $k = 0$, in the plane case there is a quite complete result on uniqueness due to Alessandrini and Valenzuela [A1V] and to Kim and Seo [KiS]. Let $\Gamma_1, \ldots, \Gamma_3$ be three connected curves with disjoint interiors whose union is $\partial\Omega$. Let $\phi_j$ be $L_2(\partial\Omega)$-functions, $\phi_j \geq 0$, and $\operatorname{supp}\phi_j \subset \Gamma_j$ and let the integral of $\phi_j$ over $\partial\Omega$ be 1. Define $g_{1,j} = \phi_3 - \phi_j$, $j = 1, 2$. In the next Theorem a buried crack $\sigma_m$ is assumed to be the union of finitely many connected nonintersecting Lipschitz curves. Let $u_{jm}$ be the solution to the boundary value problem (4.6.1) with $\sigma = \sigma_m$ and $g_1 = g_{1,j}$, $j, m = 1, 2$.

**Theorem 4.6.1.** *Let $k = 0$. Let $\Omega$ be a simply connected bounded domain in $\mathbb{R}^2$ with $C^2$-piecewise smooth Lipschitz boundary.*

*If for solutions $u_{jm}$ to the boundary value problem* (4.6.1) *one has*

$$u_{j1} = u_{j2} \text{ on } \Gamma \subset \partial\Omega, \quad j = 1, 2, \tag{4.6.2}$$

*then $\sigma_1 = \sigma_2$.*

We will not prove this result, referring instead to the above-mentioned papers. We observe only that in those proofs based on study of level-lines solutions of elliptic equations, it is used that due to the particular choice of the Neumann boundary data, solutions have no critical points inside $\Omega$, and that level curves corresponding to two boundary data form a global coordinates in $\Omega$.

We emphasize that one boundary measurement ($g_1 = g_{1,1}$) is not sufficient. A simple counterexample ( Friedman and Vogelius [FV]) is given by level curves of a harmonic function in $\Omega$. In particular, when $u(x) = x_1$ all interval of vertical straight lines can be viewed as insulating cracks and all intervals of horizontal lines as conducting cracks. A conditional logarithmic type stability estimate for $\sigma \in C^{1+\lambda}$ in terms of differences of the boundary data $u_{\sigma(1)} - u_{\sigma(2)}$ and under additional natural a priori constraints on $\sigma$ (and $\Omega$) was obtained by Alessandrini and more generality by Rondi [Ro].

An useful orthogonality relation that helps to identify linear cracks in $\mathbb{R}^2$ and planar cracks in $\mathbb{R}^3$ was found and exploited by Andrieux and Ben Abda [AnB]. We illustrate this relation in the following exericise.

**Exercise 4.6.2.** Show that for any $H_{(1)}(\Omega)$ solution $v$ to the Helmholtz equation $\Delta v + k^2 v = 0$ in $\Omega$ and a solution $u \in H_{(1)}(\Omega)$ to the boundary value problem

(4.6.1) (for insulating cracks) one has the following integral relation

$$\int_\sigma [u]\partial_\nu v d\sigma = \int_{\partial\Omega} (v g_1 - u\partial_\nu v) d\Gamma$$

where $[u]$ ("opening gap") is the difference of limits of $u$ on $\sigma$ from the "negative" and the "positive" sides of $\sigma$ determined by the unit normal $\nu$.

To derive the identity of this exercise surround $\sigma$ by "thin" domains and pass to the limit when "thinness" goes to zero.

The integral relation from Exercise 4.6.2 can be used to evaluate size of $\sigma$ by taking as $v$ particular explicit solutions to the Helmholtz equation in $\Omega$. For example, when $k = 0$ one can exploit the coordinate harmonic functions $v(x) = x_j$ Of course, when the "opening gap" is zero this relation is useless, so it is an important (and largely open) problem to find sets of boundary data $g_1$ so that one of this sets maximises this gap.

$\partial\sigma$ (end points in the plane case) are expected to be branch points of solutions to the equation (4.6.1), moreover, the solution near these points in a generic case should have special singular behavior (for example, $\nabla u(x)$ near crack tip $x_0$ is most likely $c|x - x_0|^{-1/2} + \cdots$ where $\ldots$ is bounded terms). For an effective solution of the inverse problem it would be very helpful to use the boundary data $g_1$ which maximize $c$. Again, finding such boundary data is an quite interesting, but open problem. In the plane case for $k = 0$ one can probably utilize index (winding number) of gradient of a harmonic function as in [AlIP], [AlM].

A iterative computational algorithm to find an interval $\sigma$ was suggested and tested by Santosa and Vogelius [SaV2], who actually used results of many boundary measurements with the boundary data updated on each iteration of their method.

We observe that the methods of all mentioned papers do not work for the Helmholtz equation, which models prospecting cracks by (ultrasonic or elastic) waves, and uniqueness results are unknown in the case of single or finitely many measurements. Kress [Kres] proved uniqueness of $\sigma$ from many boundary measurements (i.e., from the operator $g_1 \to g_0$ on $\Gamma$) in the plane case by the methods described in Section 3.3. He also designed an effective numerical algorithm.

In the three-dimensional case quite complete uniqueness results for insulating cracks with two boundary measurements were reported by Alessandrini and DiBenedetto [A1D]. They also proved uniqueness of planar conducting cracks. Eller [Ell] proved uniqueness of a general conducting crack in $\mathbb{R}^3$ from many boundary measurements.

Another type of cracks quite important in applications is *surface* cracks $\sigma$ consisting of finitely many connected Lipschitz components intersecting $\partial\Omega$. Elcrat, Isakov, and Neculoiu [ElcIN] proved uniqueness of such insulated cracks from one boundary measurement in a quite general situation, in particular for finitely multiconnected domains, and suggested an efficient numerical algorithm based on Schwarz-Christoffel transformations. We will formulate one result from their paper and outline its proof.

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with the $C^2$-piecewise smooth Lipschitz boundary $\partial\Omega$. Let $\Gamma_0$ be the connected component of $\partial\Omega$ that is also the boundary of the unbounded component of $\mathbb{R}^2\backslash\overline{\Omega}$. We break $\Gamma_0$ into three components, $\Gamma$, $\Gamma_1$, and $\Gamma_2$. We will consider cracks $\sigma$ satisfying the same conditions as in Theorem 4.6.1, but in contrast assuming that any component of $\sigma$ has an endpoint on $\partial\Omega\backslash(\Gamma\cup\Gamma_1)$. Also, in the boundary value problem (4.6.1) we will replace the Neumann condition on $\partial\Omega$ by the Dirichlet condition $u = g$ on $\partial\Omega$.

**Theorem 4.6.3.** *Let the Dirichlet data $g_0 \geq 0$ on $\partial\Omega$ be not identically zero, and* supp $g_0 \subset \Gamma_1$.

*Then the additional Neumann data $\partial_\nu u$ on $\Gamma$ uniquely determine $\sigma$.*

PROOF. Let us assume the opposite. Let $\sigma_1, \sigma_2$ be two collections of Lipschitz curves $\sigma_{11}, \ldots, \sigma_{1k}, \sigma_{21}, \ldots, \sigma_{2m}$ generating the same Neumann data on $\Gamma$.

Let $\Omega\backslash(\sigma_1 \cup \sigma_2)$ be not connected. Take the connected component $\Omega_0$ of this set whose boundary contains $\Gamma$. Since the two solutions $u_1, u_2$ for $\sigma_1, \sigma_2$ have the same Cauchy data on $\Gamma$, by uniqueness in the Cauchy problem for the Laplace equation they are equal on $\Omega_0$. We consider $\Omega_1 = (\Omega\backslash\sigma_1)\backslash\overline{\Omega_0}$. We have $\Delta u_1 = 0$, and we will show that $u_1 = 0$ on $\partial\Omega_1$, provided that $\Gamma_1$ is disjoint from $\partial\Omega_1$.

Let $x \in \partial\Omega_1$. If $x \in \partial(\Omega\backslash\sigma_1)$, then $u_1(x) = 0$. If not, then near $x$ there are points of $\Omega_0$ where $u_1 = u_2$; so $u_1(x) = 0$, because $x \in \sigma_2$ where $u_2 = 0$.

By the maximum principle, $u = 0$ in $\Omega_1$. Due to connectedness of $\Omega\backslash\sigma$ we have $u = 0$ in $\Omega\backslash\sigma$, which contradicts our assumption about $g$.

If $\Gamma_1$ intersects $\partial\Omega_1$, then it does not intersect $\partial\Omega_0$, and we can repeat the previous argument with $\Omega_0$ instead of $\Omega_1$.

This contradiction shows that the set $\Omega\backslash(\sigma_1 \cup \sigma_2)$ is connected. By uniqueness in the Cauchy problem $u_1 = u_2$, so $\sigma_1$ must coincide with $\sigma_2$ because $u_j = 0$ on $\sigma_j$ and $u_j > 0$ on $\Omega\backslash\sigma_j$ by the maximum principle.

The proof is complete. □

This proof is valid in three-dimensional space. In $\mathbb{R}^2$, by using harmonic conjugates, one can obtain uniqueness for conducting surface cracks.

A solution $u$ to the above direct problem can be represented by a single or double layer potential with some density distributed over $\sigma$. When this density is given, there are uniqueness and (logarithmic type) stability results for $\sigma$. We refer to the book [Is4] and to the paper by Beretta and Vessella [BerV], where this problem is related to the inverse problem of cardiology.

The boundary value problem (4.6.1) is a simplest mathematical model of a crack. A way to derive it is to approximate $\sigma$ by slender domains $D_\delta$ surrounding crack, prescribing a natural boundary condition on $\partial D_\delta$ and finding the limit of solutions for $D_\delta$ as $\delta$ goes to 0. In section 4.5 it was mentioned that limits of solutions can be different from what one expects. It is obvious that this limit depends on the choice of approximations $D_\delta$ and it is not obvious that the choice in section 4.5 was natural. Indeed, probably this approximation process repeats dynamics of the growth of crack, so it is hard to believe that $D_\delta$ must be a "uniform" layer used in section 4.5. However, it is not obvious that uniform layers are the most appropriate ones. Due to these ambiguities a more appropriate model of a stationary wave in a domain $\Omega$ with crack $\sigma$ is given by the boundary value problem

$$(A + k^2)u = 0 \text{ in } \Omega \setminus \sigma, \ \partial_{\nu(a)}u = g_1 \text{ on } \partial\Omega \tag{4.6.3}$$

with the general transmission conditions

$$u^+ = b_0 u^-, \ \partial_\nu u^+ = b_1 \partial_\nu u^- + b_{00} u^+ \text{ on } \sigma \tag{4.6.4}$$

where $u^+, u^-$ are limits of $u$ from different sides of $\sigma$ and $b_0, b_1, b_{00}$ are some coefficients which are to be found together with $\sigma$ from certain collection of boundary measurements. Here $A$ is a general elliptic operator of second order. While this general formulation is physically motivated, the particular case of $A = \Delta$ is far from understanding. In particular, it is not known whether the Neumann-to Dirichlet operator ( i.e. all possible Cauchy data for solutions $u$ to (4.6.3), (4.6.4)) uniquely determines $\sigma$ and coefficients of the transmission condition.

Few large cracks considered above hypothetically result from a collection of many microcracks. Evaluation of amount of these microcracks is quite important engineering problem. However, there are only first mathematical results available. To illustrate difficulties and features of this problem we briefly decribe the findinds of Bryan and Vogelius [BrV] for a very particular case of a periodic array of simplest possible cracks in the plane.

Let $\sigma$ be an interval in the unit square $Y = (0, 1) \times (0, 1)$ in the plane $\mathbb{R}^2$. Let $\sigma$ has the end points $(s, s), (1 - s, 1 - s), 0 < s < 1/2$. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^2$-boundary. We denote by $\sigma(\varepsilon)$ the union of $\varepsilon$-scaled translations $\varepsilon((n_1, n_2) + \sigma)$ of $\sigma$ over all integer $n_1, n_2$. Let $\Omega(\varepsilon) = \Omega \setminus \sigma(\varepsilon)$. An electric potential $u(;\varepsilon)$ in the domain $\Omega(\varepsilon)$ with periodic array of small insulating cracks $\sigma(\varepsilon)$ solves the following Neumann problem for the Laplace equation:

$$\Delta u(;\varepsilon) = 0 \ \text{ in } \Omega(\varepsilon), \ \partial_\nu u(;\varepsilon) = 0 \text{ on } \sigma(\varepsilon) \cap \Omega, \ \partial_\nu u(;\varepsilon) = g_1 \text{ on } \partial\Omega,$$

where we assume that the total flux ( the integral of $g_1$ over $\partial\Omega$) is zero and for uniqueness of $u(;\varepsilon)$ we request one of standard normalization conditions, for example assuming that the integral of $u(;\varepsilon)$ over $\Omega$ is zero. Because of expected complex behavior of $u(;\varepsilon)$ as $\varepsilon$ goes to zero one would be interested in finding the limit (in some sense) of $u(;\varepsilon)$. The most suitable available technique is the so-called homogenization, also used in [BrV]. Homogenization is rather complicated procedure and it is not quite clear how to effectively utilize it to solve inverse problems.

To continue with results of [BrV] we introduce the $(y_1, y_2)$-periodic ( with periods $(1, 0), (0, 1)$) solution $\chi_k(y)$ to the boundary value problem

$$\Delta \chi_k = 0 \text{ in } \mathbb{R}^2 \setminus \sigma(1), \ \partial_\nu \chi_k = -\nu_k \text{ on } \sigma(1)$$

which is unique up to an additive constant. Let $a$ be the positive symmetric matrix

$$a_{jk} = \int_Y (\delta_{jk} + \partial \chi_j / \partial y_k)$$

and let $u$ be the normalized solution to the Neumann problem

$$div(a\nabla u) = 0, \text{ in } \Omega, \ \partial_{\nu(a)}u = g_1 \text{ on } \partial\Omega.$$

One of typical results obtained by homogenization methods (use of two-scale test oscillating test function) claims that in certain sense $u(;\varepsilon)$ converges to $u$ as $\varepsilon$ goes to 0.

Another related problem is about prospecting corrosion on inaccessible parts of surfaces. A model of electric prospecting is described by the elliptic boundary value problem

$$\Delta u = 0 \text{ in } \Omega_\gamma, \quad \left(\int_\Omega u = 0, \text{ when } b = 0\right)$$
$$\partial_\nu u + bu = g_1 \text{ on } \partial\Omega, \quad \left(\int_\Gamma g_1 = 0, \text{ when } b = 0\right),$$

where $\Omega_\gamma$ is the domain $\{x : 0 < x_1 < 1, \ldots, 0 < x_{n-1} < 1, 0 < x_n < \gamma(x')\}$ and $\Gamma_0$ is the part $\partial\Omega \cap \{x_n = 0\}$ of its boundary. It is assumed that $g_1 = 0$ on $\partial\Omega\backslash\Gamma_0$, $0 \leq b$, and $b = 0$ outside $\partial\Omega \cap \{x_n = \gamma(x')\}$ (the possibly corroded boundary part). One is looking for functions $\gamma$ and $b$ (or a more general nonlinear boundary condition) from the additional Dirichlet data

$$u = g_0 \quad \text{on } \Gamma_0.$$

When $g_1$ is not identically zero and $b = 0$ is given, the uniqueness of $\gamma$ can be shown as for obstacles in Section 6.3. One boundary measurement is obviously not sufficient to determine both $\gamma$ and $b$, so it makes sense to consider two measurements (or even the local Neumann-to-Dirichlet map on $\Gamma_0$ especially when one tries to determine a nonlinear boundary condition ($b = b(x, u)$). While this inverse corrosion problem is mathematically largely open, there are preliminary uniqueness theorems and numerical reconstruction algorithms.

We will conclude with a quite explicit counterexample of Alessandrini [Al4] which shows exponential instability of the inverse problem about determination $\gamma$ from one set of remote data on $\Gamma_0$

COUNTEREXAMPLE 4.6.4. Let $S_m$ be the strip in $\mathbb{R}^2$ bounded by the $x_1$-axis $\Gamma_0$ and the curve $\gamma_m$ given by the parametric equations $x_1 = t + 1/(2m)sinmt, x_2 = 1 + 1/(2m)(1 - e^{-2m})/(1 + e^{2m})cosmt, t \in \mathbb{R}, m = 1, 2, \ldots$. Let $S_0$ be the strip bounded by the curves $\Gamma_0$ and $\gamma = \{x_2 = 1\}$. For the (Hausdorff) distance $dist(\gamma_m, \gamma)$ between the curves $\gamma_m$ and $\gamma$ we have

$$1/(4m) \leq dist(\gamma_m, \gamma) \leq 1/(2m).$$

These inequalities can be derived by using the definition of the Hausdorff distance and minimization and maximization of functions of one variable. We will consider the solutions $u_m$ to the following Neumann problem

$$\Delta u_m = 0 \text{ in } S_m$$

$$\partial_\nu u_m = 0 \text{ on } \gamma_m \cap \Gamma_0 \tag{4.6.5}$$

with the source type condition at infinity

$$u_m(x) - x_1 = O(1)$$

where $O(1)$ is bounded as $|x|$ goes to infinity. We are interested in finding $\gamma$ from the Dirichlet data

$$u_m = g_{0,m} \text{ on } \Gamma_0.$$

We will use complex variable $z = x_1 + ix_2$. The complex analytic function

$$z_m(w) = w + 1/(2mcoshm)sinmw$$

is a conformal mapping of the strip $S_0$ onto $S_m$. Let $w_m(z)$ be the inverse mapping. One can check that the function

$$u_m(z) = \Re(w_m(z))$$

solves the Neumann problem (4.6.5) and satisfies the conditions at infinity. It is obvious that $u_0(z) = x_1$ solves the same problem in $S_0$. Let $g_m, g_0$ be the Dirichlet data for $u_m, u_0$ on $\Gamma_0$. From the definitons, $g_{0,m}((x_1, 0)) + 1/(m(e^m + e^{-m})sin(g_{0,m}(x_1, 0))) = x_1$, $g_{0,0}((x_1, 0)) = x_1$. Hence,

$$\|g_{0,m} - g_{0,0}\|_\infty(\Gamma_0) \le e^{-m}$$

while the distance between curves $\gamma_m$ and $\gamma$ is not less than $1/(4m)$. This shows exponential instability for this inverse problem. Given any $l$ by small modification one can obtain similar examples when $\gamma_m$ are the graphs of functions with bounded $C^l(\mathbb{R})$-norms.

To obtain a counterexample with bounded domains $\Omega_m, \Omega_0$ instead of $S_m, S_0$ one tranplants the Neumann problem with the aid of the conformal mapping

$$z(w) = (e^{\pi/2w} - 1)/(e^{\pi/2w} + 1)$$

transforming the strip $S_0$ in the $w$-plane into the unit upper half-disk $\Omega_0$ in the $z$-plane, and $S_m$ into small perturbations $\Omega_m$ of $\Omega_0$. Observe that the image of $\Gamma_0$ is the interval $\Gamma_0^*$ with the endpoints $-1$, 1. The Neumann problems (4.6.5) will be transformed into the Neumann problems in $\Omega_m$ and the the condition at in infinity will take the form

$$\partial_\nu u_m = \delta(1) - \delta(-1) \text{ on } \Gamma_0.$$

## 4.7 Open problems

We list some quite important and difficult questions with few comments.

**Problem 4.1.** Prove uniqueness in the inverse problem of gravimetry for star-shaped $D$ when $k$ is given, positive, and Lipschitz.

There is a counterexample in [Is4], section 3.4, for $x_n$-convex $D$ when $k$ is given, positive, and Hölder continuous.

**Problem 4.2.** Prove the uniqueness of convex (better, star-shaped $D$) entering the Dirichlet problem

$$-\Delta u + \chi(D)u = 0 \quad \text{in } \Omega, \qquad u = g \text{ on } \partial\Omega$$

with the given additional Neumann data $\partial_\nu u$ on $\partial\Omega$.

A part of this problem is to determine sufficient conditions on $\Omega$ and $g$. As mentioned above, this problem has applications in semiconductor theory.

**Problem 4.3.** Obtain a global existence theorem for an unknown coefficient $c$ that does not depend on $x_n$ in the situation described in Section 4.2.

To solve this problem one can try to modify the argument for the inverse parabolic problem, with the final overdetermination considered in Section 9.1.

**Problem 4.4.** Obtain a local uniqueness result similar to Theorem 4.4.3 in the three-dimensional case.

In this case conformal mappings technique is not available. In addition, the linearized problem (oblique derivative problem when the boundary differentiation direction is tangent to the boundary) is not elliptic, so it would be extremely difficult to use only one set of boundary data. One suggestion is to use two or three sets to secure ellipticity of the resulting linearized problem. Another idea is to consider unknown symmetric (with respect to a plane or a point) domains.

**Problem 4.5.** Prove global uniqueness of convex $D$ in the inverse conductivity problem with no interior source.

A difficulty of this problem is illuminated in comments to problem 4.4. One can not repeat the proof of Theorem 4.3.7 (where the field is generated by an interior source), because any experior generation implies that the integral of $\partial_\nu u$ over $\partial D$ is zero, so we do not have positivity of the normal derivative. Uniqueness of balls from one boundary measurement was proven by Kang and Seo [KS1], [KS2] who used single layer representation.

**Problem 4.6.** Prove (global) uniqueness of both $\gamma$ and $b = b(x)$ in the inverse corrosion problem.

A part of the problem is to find two Dirichlet (or Neumann) data which guarantee uniqueness of the recovery of the surface and of the boundary coefficient. In a sense, these data must be independent.

# 5

# Elliptic Equations: Many Boundary Measurements

## 5.0 The Dirichlet-to-Neumann map

We consider the Dirichlet problem (4.0.1), (4.0.2). We assume that for any Dirichlet data $g$ we are given the Neumann data $h$; in other words, we know the results of all possible boundary measurements, or the so-called Dirichlet-to-Neumann operator $\Lambda : H_{(1/2)}(\partial\Omega) \to H_{(-1/2)}(\partial\Omega)$, which maps the Dirichlet data $g_0$ into the Neumann data $g_1$. From Theorem 4.1 the operator $\Lambda$ is well-defined and continuous, provided that $\Omega$ is a bounded domain with Lipschitz $\partial\Omega$. In Sections 5.1, 5.4, 5.7 we consider scalar $a$, $b = 0$, $c = 0$. The study of this problem was initiated by the paper of Calderon [C], who studied the inverse problem linearized around a constant and suggested a fruitful approach, which was extended by Sylvester and Uhlmann in their fundamental paper [SyU2], where the uniqueness problem was completely solved in the three-dimensional case.

We will consider two equations with the coefficients $a_1, b_1, c_1$ and $a_2, b_2, c_2$. We will denote the corresponding Dirichlet-to-Neumann maps by $\Lambda_1$, $\Lambda_2$, and solutions of these equations by $u_1, u_2$. Due to substantial use of complex-valued solutions and further relations with scattering theory, we recall the definition of a weak solution in the complex case, considering complex-valued coefficients $b$, $c$ (but only real-valued $a$).

The definition (4.0.3) of a generalized solution $u \in H_{(1)}(\Omega)$ of the equation $-\operatorname{div}(a\nabla u) + b \cdot \nabla u + cu = 0$ in $\Omega$ reads

$$\int_\Omega \left(a\nabla u \cdot \overline{\phi} + b \cdot \nabla u\overline{\phi} + cu\overline{\phi}\right) = \int_{\partial\Omega} \Lambda u\overline{\phi} \tag{5.0.1}$$

for any test function $\phi \in H_{(1)}(\Omega)$. Letting $\phi = u^*$ in (5.0.1), where $u^*$ is a solution to the adjoint equation

$$-\operatorname{div}(a\nabla u^*) - \overline{b} \cdot \nabla u^* + (\overline{c} - \operatorname{div}\overline{b})u^* = 0 \text{ in } \Omega$$

and taking the complex conjugate of this integral relation, we obtain

$$\int_{\Omega}(a\nabla\overline{u}\cdot\nabla u^* + \overline{b}\cdot\nabla\overline{u}u^* + \overline{cu}u^*) = \int_{\partial\Omega}\overline{\Lambda u}u^*.$$

Using that for the symmetric matrix $a$ we have $a\nabla\overline{u}\cdot\nabla u^* = a\nabla u^*\cdot\nabla\overline{u}$ and integrating by parts to refer $\nabla$ to $u^*$, we get

$$\int_{\Omega}\big(a\nabla u^*\cdot\nabla\overline{u} - \overline{b}\cdot\nabla u^*\overline{u} + (\overline{c} - \operatorname{div}\overline{b})u^*\overline{u}\big) = \int_{\partial\Omega}u^*(\overline{\Lambda u} - \overline{b}\cdot\nu\overline{u}).$$

On the other hand, the definition (5.0.1) of a generalized solution $u^*$ to the adjoint equation with the test function $u$ gives that the left side is equal to the integral over $\partial\Omega$ of $\Lambda_* u^*\overline{u}$, where $\Lambda_*$ denotes the Dirichlet-to-Neumann map of the adjoint equation. Recalling that

$$(g_0, g_1)_2(\partial\Omega) = \int_{\partial\Omega}g_0\bar{g}_1$$

denotes the scalar product in the Hilbert space of complex-valued functions $L_2(\partial\Omega)$, we obtain

$$(u^*, \Lambda u - b\cdot\nu u)_2(\partial\Omega) = (\Lambda_* u^*, u)_2(\partial\Omega)$$

for any solution $u$ to the first equation and any solution $u^*$ to its adjoint. Due to our assumptions, the Dirichlet problem for these equations is uniquely solvable for any boundary data in $H_{(1/2)}(\partial\Omega)$. Since this space is dense in $L_2$, we conclude that

$$\Lambda_* = \Lambda^* - \overline{b}\cdot\nu \tag{5.0.2}$$

Observe that when $\Re b = 0$, $\operatorname{div} b = 0$ and $\Im c = 0$, our equation is self-adjoint, so when in addition $b\cdot\nu = 0$ on $\partial\Omega$, one can conclude that the Dirichlet-to-Neumann operator $\Lambda$ is self-adjoint.

The definition (5.0.1) and the relation (5.0.2) imply a fundamental orthogonality relation as follows. The definition (5.0.1) of a solution $u_1$ to the first equation with the test function $u_2^*$ is

$$\int_{\Omega}\big(a_1\nabla u_1\cdot\nabla\overline{u}_2^* + b_1\cdot\nabla u_1\overline{u}_2^* + c_1u_1\overline{u}_2^*\big) = \int_{\partial\Omega}(\Lambda_1u_1)\overline{u}_2^*,$$

and the same definition for the adjoint of the second equation with the test function $u_1$ gives

$$\int_{\Omega}\big(a_2\nabla u_2^*\cdot\nabla\overline{u}_1 - \overline{b}_2\cdot\nabla u_2^*\overline{u}_1 + (\overline{c}_2 - \operatorname{div}\overline{b}_2)u_2^*\overline{u}_1\big)$$
$$= \int_{\partial\Omega}(\Lambda_{2*}u_2^*)\overline{u}_1.$$

Subtracting the first equality from the complex adjoint of the second equality, and using integration by parts

$$\int_{\Omega} -b_2\cdot\nabla\overline{u}_2^*u_1 = \int_{\partial\Omega} -b_2\cdot\nu\overline{u}_2^*u_1 + \int_{\Omega}(b_2\cdot\nabla u_1\overline{u}_2^* + \operatorname{div} b_2u_1\overline{u}_2^*),$$

relation (5.0.2) for $\Lambda = \Lambda_2$, and the definition of the adjoint operator in $L_2(\partial\Omega)$, we obtain

$$\int_\Omega ((a_2 - a_1)\nabla u_1 \cdot \nabla \overline{u}_2^* + (b_2 - b_1)\cdot \nabla u_1 \overline{u}_2^*) + (c_2 - c_1)u_1\overline{u}_2^*$$

$$= \int_{\partial\Omega} (\Lambda_2 - \Lambda_1)u_1\overline{u}_2^* \tag{5.0.3}$$

for any solution $u_1 \in H_{(1)}(\Omega)$ to the first equation and for any solution $u_2^*$ to the adjoint to the second equation. When $\Re b_2 = 0$, $\operatorname{div} b_2 = 0$, and $\Im c_2 = 0$, the second equation is self-adjoint, so we can drop * over $u_2$. When $b_1 = b_2 = 0$, then $\overline{u}_2^*$ solves the equation $\operatorname{div}(a_2\nabla u_2) + c_2 u_2 = 0$, and from (5.0.3) we have the identity first established and used by Alessandrini [Al1] under the additional assumption $\Im c_j = 0$.

$$\int_\Omega (a_2 - a_1)\nabla u_1 \cdot \nabla u_2 + (c_2 - c_1)u_1 u_2 = \int_{\partial\Omega} ((\Lambda_2 - \Lambda_1)u_1)u_2 \tag{5.0.4}$$

for all solutions $u_1, u_2$ in $H_{(1)}(\Omega)$.

**Exercise 5.0.** Prove the identity

$$\int_\Omega ((a-1)\nabla u \cdot \nabla v + b\nabla u \cdot v + cuv) = \int_{\partial\Omega} ((-\Lambda_0 + \Lambda(f))u)v + \int_\Omega fv$$

for any solution $u \in H_{(1)}(\Omega)$ to the inhomogeneous equation $-\operatorname{div}(a\nabla u) + b \cdot \nabla u + cu = f$ in $\Omega$ and any harmonic function $v \in H(1)(\Omega)$. Here $\Lambda(f)$ is the Dirichlet-to-Neumann map for the inhomogeneous differential equation with right side $f$.

Consider the case $a = 1, b = 0$. Letting $v = 1$ and approximating by $f$ the Dirac-delta function with the pole at $x \in \Omega$, derive that

$$\int_\Omega cG(x, \ ) = F(x) + 1, \tag{5.0.5}$$

where $G$ is Green's function of the Dirichlet problem for the equation $-\Delta u + cu = 0$ and

$$F(x) = \int_{\partial\Omega} \partial_\nu G(x, \ )$$

is uniquely determined in a stable way by $\partial_\nu u$ on $\partial\Omega$ given for all $f$. Applying the operator $-\Delta + c$ to both sides of (5.0.5), prove that

$$c(x) = \Delta F(x)/F(x), \tag{5.0.6}$$

provided that $c \geq 0$ in $\Omega$.

{*Hint*: to derive the initial formula, repeat the proof of formula (5.0.3) for the inhomogeneous first equation. Let $a_2 = 1$, $b_2 = 0$, $c_2 = 0$. Make use of the assumption $c \geq 0$ and of Giraud's maximum principle to guarantee that $F < 0$ when $x \in \Omega$.}

## 5.1 Boundary reconstruction

The first (and simplest) step in the reconstruction of $a$ is finding $a$ on $\partial\Omega$. In this section we consider the scalar conductivity coefficient $a$.

**Theorem 5.1.1** (Uniqueness). *Assume that $a_1, a_2 \in C^k(V)$, where $V$ is a neighborhood of a boundary point of $\Omega$.*

*If $\Lambda_1 = \Lambda_2$, then $\partial^\alpha a_1 = \partial^\alpha a_2$ on $\partial\Omega \cap V$ when $|\alpha| \le k$.*

A first version of this result was obtained by Kohn and Vogelius in their paper [KoV1], where they assumed that $a_j \in C^\infty$ and first used rapidly oscillating solutions in the so-called orthogonality relations (see below). A constructive method was suggested by Nachman ([N1], Theorem 1.5). He proved that $2e^{-ix\cdot\xi} S\Lambda_a e^{ix\cdot\xi}$ converges to $a(x)$ when $|\xi|$ goes to $+\infty$. Here $S$ is the classical single layer potential operator (for the Laplace equation). The version above belongs to Alessandrini [Al3]. In this section we consider scalar $a$ and real-valued solutions $u$.

PROOF OF THEOREM 5.1.1 FOR $k = 0, 1$. We first apply the formula (5.0.4) and obtain the orthogonality relations

$$\int_\Omega (a_2 - a_1)\nabla u_1 \cdot \nabla u_2 = 0 \tag{5.1.1}$$

for all solutions $u_1, u_2$ to the equations $\operatorname{div}(a_j \nabla u_j) = 0$ near $\overline{\Omega}$. For any $m = 0, 1, 2 \ldots$ Alessandrini [Al3] constructed the so-called singular solutions $u_j(x, y)$ to equation (4.0.1) with $a \in C^1$ with a pole at $x$ having the following properties:

$$u_j(x, y) = K(x^* - y^*)S_m((x^* - y^*)/|x^* - y^*|) + w(x, y) \tag{5.1.2}$$

when $x \ne y$, where $K(x)$ is $\ln|x|$ when $n = 2$, $m = 0$, and $|x - y|^{2-n-m}$ otherwise; $z^* = A^*(x)z$, where $A^*(x)$ is a linear operator in $\mathbb{R}^n$ transforming the principal part of $A$ into the Laplacian at a point $x$; $S_m$ is the spherical harmonic of degree $m$, and $|w| + |x - y||\nabla_y w| \le C|x - y|^{5/2-n-m}$. It is easy to observe that $K(x)S_m(x/|x|)$ is a partial derivative of order $m$ of the classical fundamental solution $K(x)$ of the Laplace operator. When $m = 0, 1$ and $a \in C^{1+\lambda}$, existence of such singular solutions follows from the classical construction of fundamental solutions of second-order elliptic equations described in the book of Miranda ([Mi], sections 8, 15, 19).

To show that $a_1 = a_2$ on $\partial\Omega \cap V$ we assume the opposite. Then we may assume that the point $0 \in \partial\Omega \cap V$ and $a_1(0) < a_2(0)$. Letting $m = 0$ we obtain singular solutions $u_1, u_2$ such that $|\nabla_y u_1 \cdot \nabla_y u_2| \ge \varepsilon |x - y|^{2-2n}$ when $x, y$ are contained in a ball $B$ of small radius centered at the origin. We may assume also that $a_2 - a_1 > \varepsilon$ on $B$. When $x \in B\setminus\overline{\Omega}$, the function $u_j(x, \ )$ is a solution to equation $\operatorname{div} a_j \nabla u = 0$, so from the orthogonality relations (5.1.1) we have

$$-\int_{\Omega\setminus B}(a_2-a_1)\nabla u_1(x,)\cdot\nabla u_2(x,)=\int_{B\cap\Omega}(a_2-a_1)\nabla u_1(x,)\cdot\nabla u_2(x,)$$
$$\geq \varepsilon^2\int_{B\cap\Omega}|x-y|^{2-2n}dy.$$

Since the singularity $x$ is in $B$, the left-side integral is bounded when $x$ goes to the origin. On other hand, it is easy to see that the right-side integral is unbounded for these $x$. We have obtained a contradiction, which shows that our assumption was wrong. So $a_1=a_2$ on $\partial\Omega\cap V$.

Let us consider the case $k=1$. As shown, $a_1=a_2$ on $\partial\Omega\cap V$. Let us assume that the first derivatives of $a_1, a_2$ do not agree on this set. Observe that a Lipschitz surface $\partial\Omega\cap V$ has a tangential hyperplane almost everywhere (see the book of Morrey [Mor], Theorem 3.1.6). Since the coefficients agree on the boundary, their tangential derivatives agree, so due to our conjecture, we can assume that the point $0\in\partial\Omega\cap V$ and $\partial_\nu(a_2-a_1)(0)>\varepsilon>0$. We can then assume that the direction of the interior normal to $\partial\Omega$ at 0 coincides with the direction of the $x_1$-coordinate axis. Since $\partial\Omega$ is Lipschitz, the angles between the interior normals and the $x_1$-directions are smaller than $\pi/2-\varepsilon_1$. Using that tangential components of $\nabla a_j$ agree and that these gradients are continuous at 0, we conclude that $\partial_1(a_2-a_1)(x)>0$ for all $x\in\partial\Omega$ near the origin. Since $a_2-a_1=0$ on $\partial\Omega$ near 0, from Taylor's formula we conclude that $a_2-a_1>0$ on $\Omega$ near 0.

If $n\geq 3$ let $u_j(y)=K(x-y)+w_j$ in (5.1.2) ($m=0$). Then direct calculations show that $\nabla_y K(x-y)\cdot\nabla_y K(x-y)=C|x-y|^{2-2n}$, so from the estimates for $w_j$ we have

$$\nabla u_1\cdot\nabla u_2\geq\varepsilon|x-y|^{2-2n}\text{ when }|x-y|<\varepsilon$$

If $n=2$, we let $m=1$ in (5.1.2) and $u_j(y)=C\partial_2K(x-y)+w_j$. Direct calculations show that

$$\nabla\partial_2K(x-y)\cdot\nabla\partial_2K(x-y)=|x-y|^{-8}(x_1^4-x_1^2x_2^2+x_2^4)\geq 2^{-1}|x-y|^{-4}.$$

So as above, $\nabla u_1\cdot\nabla u_2\geq\varepsilon|x-y|^{-4}$ when $|x-y|<\varepsilon$.

Now we use the scheme of proof with $k=0$. Let $B$ be the ball $B(0;\varepsilon/2)$ and let $x=(-\tau,0,\ldots,0)$, where $0<\tau<\varepsilon/4$. Since the hyperplane $\{x_1=0\}$ is tangent to $\partial\Omega$ at 0, there is a solid cone $\mathcal{C}=\{|y/|y|-e_1|<\varepsilon, |y|<\varepsilon\}$ that is contained in $\Omega$ when $\varepsilon$ is small. Observe that $|x-y|\geq\varepsilon/4$ when $y\in\Omega\setminus B$.

Putting all this together, we obtain

$$-\int_{\Omega\setminus B}(a_2-a_1)\nabla u_1\cdot\nabla u_2=\int_{\Omega\cap B}(a_2-a_1)\nabla u_1\cdot\nabla u_2$$
$$\geq\varepsilon\int_{\mathcal{C}}y_1|x-y|^{-l}dy,$$

where $l = 2n - 2$ when $n \geq 3$ and $l = 4$ when $n = 2$. As above, the left integral is bounded with respect to $x$, so the integral

$$\int_{\mathcal{C}} y_1 |x - y|^{-l} dy$$

must be bounded as well. Then, by using the Lebesgue dominated convergence theorem, we obtain that the integral of $|y|^{1-l}$ over $\mathcal{C}$ is convergent, which is false (make use of polar coordinates). The contradiction shows that the gradients of $a_1$ and $a_2$ coincide on $\partial\Omega \cap V$.

Similarly, by using singular solutions with larger $m$ one can complete the proof for all $k$. □

We observe that this proof is valid when $k = 0, 1$, and also for the more general equation with $b_1 = b_2 = 0$.

The claim is generally false for the anisotropic case.

**Exercise 5.1.2**. Show that

$$\int_{\mathcal{C}} |x - y|^{-k} dy \geq \left\{ \begin{pmatrix} -\varepsilon \ln |x| & \text{when } k = n, \\ \varepsilon |x|^{n-k} & \text{when } k > n. \end{pmatrix} \right.$$

{*Hint*: Make use of polar coordinates, the triangle inequality, and substitutions.}

Practically, the results of measurements are not always available for the whole boundary, so the following local version of the inverse problem is interesting.

Let $\Gamma$ be an open part of $\partial\Omega$. Denote by $\Lambda_\Gamma$ the local Dirichlet-to-Neumann map that is defined for functions $g \in H_{(1/2)}(\partial\Omega)$ that are zero on $\Omega \backslash \Gamma$ and that maps them into the conormal derivative of the solution $a\nabla u \cdot \nu$ on $\Gamma$.

**Exercise 5.1.3**. Prove that if $a \in C^k(V)$, where $V$ is a neighborhood of $\Gamma$, then even $\Lambda_\Gamma$ uniquely determines $\partial^\alpha a$ on $\Gamma$ when $|\alpha| \leq k$.

{*Hint*: In the proof of Theorem 5.1.1 use as $u(x, )$ Green's function of the Dirichlet problem for the elliptic equation in the domain $\Omega^*$ that contains $\Omega$, such that $\partial\Omega \backslash \Gamma$ is in $\partial\Omega^*$. Let $x \in \Omega^* \backslash \overline{\Omega}$}

**Exercise 5.1.4**. Derive the relation

$$a \int_{\partial\Omega} u\nu_1 = \int_{\partial\Omega} (\Lambda u)v,$$

where $u$ is a solution of the conductivity equation with the constant coefficient $a$ and $v(x) = x_1$.

When the integral of $u\nu_1$ over $\partial\Omega$ is not zero (say, $u > 0$ when $\nu_1 > 0$ and $u < 0$ when $\nu_1 < 0$), one can use this formula for determination of the constant $a$.

Singular solutions allowed Alessandrini [Al3] to obtain the following estimate as well.

**Theorem 5.1.5**. *Let* $|a_j|_{k+1}(\Omega) < M$ *and* $\delta = 1$ *when* $k = 0$, *and* $(1 + 2)^{-1} \dots (1 + 2k)^{-1}$ *when* $k = 1, 2, \dots$.

*Then*

$$|a_2 - a_1|_k(\partial\Omega) \le C\|\Lambda_2 - \Lambda_1\|^\delta,$$

*where C depends only on* $\Omega, M, k$ *and* $\inf a_j$ *over* $\Omega$, *and* $\|\,\|$ *is the operator norm of* $\Lambda_2 - \Lambda_1$ *considered from* $H_{(1/2)}(\partial\Omega)$ *into* $H_{(-1/2)}(\partial\Omega)$.

PROOF FOR $k = 0, n \ge 3$. Since the $a_j$ are continuous, we can assume that $\|a_2 - a_1\|_\infty(\partial\Omega)$ is $(a_2 - a_1)(x^0) > 0$ at a point $x^0 \in \partial\Omega$. Since $\partial\Omega$ is Lipschitz, in any neighborhood of $x^0$ there are points of $\partial\Omega$ where there is a tangential hyperplane. By using continuity we can find such a point where $a_2 - a_1$ is greater than one-half of the value at $x^0$. Finally, we can assume that this point is the origin, that the coordinate system is chosen as in the proof of Theorem 5.1.1, and that

$$\|a_2 - a_1\|_\infty(\partial\Omega) \le 2(a_2 - a_1)(0) \le 2(a_2 - a_1)(x) + o(|x|).$$

Using the identity (5.0.4) and splitting the integral of $(a_2 - a_1)\nabla u_2 \cdot \nabla u_2$ as in the proof of Theorem 5.1.1, we obtain

$$\begin{aligned} &\|a_2 - a_1\|_\infty(\partial\Omega) \int_{\mathcal{C}} |x - y|^{2-2n} dy \\ &\le C\Big( \int_B |x - y|^{2-2n} o(|y|) dy + \int_{\Omega\setminus B} |\nabla u_1 \cdot \nabla u_2| \\ &\quad + \|\Lambda_2 - \Lambda_1\| \|u_1\|_{(1/2)}(\partial\Omega) \|u_2\|_{(1/2)}(\partial\Omega)\Big). \end{aligned} \tag{5.1.3}$$

We evaluate the integral over the cone $\mathcal{C}$. Observe that

$$|x - y|^2 = ((y_1 + \tau)^2 + y_2^2 + \dots + y_n^2) \le 2(|y|^2 + \tau^2)$$

and make use of polar coordinates centered at 0, $\rho = |y|$. Then

$$\begin{aligned} \int_{\mathcal{C}} |x - y|^{2-2n} dy &\ge \varepsilon_1 \int_0^\varepsilon (\rho^2 + \tau^2)^{1-n} \rho^{n-1} d\rho \\ &= \varepsilon_1 n^{-1} \int_0^{\varepsilon^n} (\sigma^{2/n} + \tau^2)^{1-n} d\sigma \ge \varepsilon_2 \int_0^{\varepsilon^n} (\sigma + \tau^n)^{2/n-2} d\sigma, \end{aligned}$$

where we used the substitution $\rho^n = \sigma$ and the inequality $(\sigma^{2/n} + \tau^2) \le C(\sigma + \tau^n)^{2/n}$. Calculating the last integral, we obtain

$$\int_{\mathcal{C}} |x - y|^{2-2n} dy \ge \varepsilon_3 \tau^{2-n} - C(\varepsilon). \tag{5.1.4}$$

Similarly, we conclude that the first integral on the right side of (5.1.3) is less than $o(\varepsilon)\varepsilon_3\tau^{2-n}$. The second integral is bounded by $C(\varepsilon)$.

To complete the proof we bound $\|u_j\|_{(1/2)}(\partial\Omega) \le C\|u_j\|_{(1)}(\Omega)$ by using trace theorems. It suffices to bound $\|\nabla u_j\|_2(\Omega)$. From the construction (5.1.2) we have $|\nabla u_j| \le C|x - y|^{1-n}$. Arguing as for the integral over $\mathcal{C}$, we conclude that

$\|\nabla u_j\|_2^2(\Omega) \le C\tau^{2-n}$. Therefore, the factor of $\|\Lambda_2 - \Lambda_1\|$ in (5.1.3) is bounded by $C\tau^{2-n}$.

Using the bound (5.1.4) and upper bounds for other terms in (5.1.3), we get

$$\|a_2 - a_1\|_\infty(\partial\Omega)(1 - \tau^{n-2}C(\varepsilon)) \le (o(\varepsilon) + \tau^{n-2}C(\varepsilon) + C\|\Lambda_2 - \Lambda_1\|).$$

Choosing $\varepsilon$ so small that $o(\varepsilon) < \|\Lambda_2 - \Lambda_1\|$ and then letting $\tau \to 0$, we complete the proof. □

The proof for $n = 2$ needs minor modifications, and the result for $k \ge 1$ can be obtained from the result for $k = 0$ by using interpolation.

**Exercise 5.1.6**. Let $|c_j|_1(\Omega) < M$. Show that

$$|c_2 - c_1|_0(\partial\Omega) \le C\|\Lambda_2 - \Lambda_1\|^{1/3}$$

where $C$ depends only on $\Omega, M$ and $\|\Lambda_2 - \Lambda_1\|$ is the operator norm from $H_{(1/2)}(\partial\Omega)$ into $H_{(-1/2)}(\Omega)$. Here $\Lambda_j$ is the Dirichlet-to Neumann operator for the equation $(-\Delta + c_j) = 0$ in $\Omega$.

{*Hint*: Repeat the proof of Theorem 5.1.5 with an appropriate choice of singular solutions $u_1, u_2$ (e.g. some first order derivatives of the Green's function) and at the end of the proof select $\tau$ as a power of $\|\Lambda_2 - \Lambda_1\|$.}

Sylvester and Uhlmann [SyU3] combined methods of multiple layer potentials and pseudodifferential operators to obtain these estimates for $k = 0, 1$ under minimal assumptions on $a_j$ but under additional smoothness assumptions on $\partial\Omega$. Say, when $k = 0$, they assume only that the $a_j$ are continuous, and then constants on their estimates depend only on upper and lower uniform bounds of $a_j$ in $\Omega$.

## 5.2 Reconstruction in $\Omega$

The most interesting and difficult part of the inverse problem is reconstruction of $a$ inside $\Omega$. We have the following global uniqueness result.

**Theorem 5.2.1**. *Let $n \ge 3$. Let the conductivity coefficient $a$ be in $H_{2,\infty}(\Omega)$.*

*Then $a$ is uniquely determined by Dirichlet-to-Neumann map for the conductivity equation* $\operatorname{div}(a\nabla u) = 0$ *in* $\Omega$.

PROOF. Suppose we have two conductivity coefficients $a_1, a_2$ with equal Dirichlet-to-Neumann maps. First, one reduces conductivity equation (4.3.1) to the Schrödinger equation

$$-\Delta u^* + c_j u_j^* = 0 \quad \text{in } \Omega, \quad c_j = a_j^{-1/2}\Delta a_j^{1/2} \tag{5.2.1}$$

by the well-known substitution $u_j = a_j^{-1/2}u_j^*$, which makes sense when $a \in H_{2,\infty}(\Omega)$. By Theorems 5.1.1 we have $a_1 = a_2$, $\nabla a_1 = \nabla a_2$ on $\partial\Omega$. So the equality of the Dirichlet-to-Neumann maps for conductivity equations (4.3.1) will imply the equality for equations (5.2.1). By Theorem 5.2.2 we have $c_1 = c_2$. The second relation (5.2.1) is an elliptic differential equation with respect to $a_j^{1/2}$ with bounded measurable coefficient $c_j$. Since $a_1^{1/2}$, $a_2^{1/2}$ satisfy the same equation in $\Omega$, and $a_1 = a_2$, $\nabla a_1 = \nabla a_2$ on $\partial\Omega$, we conclude that $a_1 = a_2$ in $\Omega$ by uniqueness in the Cauchy problem (Section 3.3). The proof is complete. $\square$

From now on we will consider equation (5.2.1) with complex-valued $c_j$ and we drop the sign*.

**Theorem 5.2.2**. *Let* $n \geq 3$. *Let* $c_j \in L_\infty(\Omega)$.

*If equations* (5.2.1) *have the same Dirichlet-to-Neumann maps, then* $c_1 = c_2$.

This result (under the assumption that $c_j \in C^\infty(\overline{\Omega})$) was obtained in the fundamental paper of Sylvester and Uhlmann [SyU2]. In fact, the results and methods of this paper are valid for $c_j \in L_\infty(\Omega)$. Nachman [N2] announced uniqueness when $c_j \in L_{n/2}(\Omega)$. So Theorem 5.2.1 is valid when $a \in H_{2,n/2}(\Omega)$. Greenleaf, Lassas, and Uhlmann [GLU] showed uniqueness for potentials $c$ which are some conormal distributions (to smooth surfaces), however these distributions do not include for example single layer distributions. Recently, Bukhgeim and Uhlmann [BuU] proved that for uniqueness in Theorems 5.2.1, 5.2.2 it suffices to know $\Lambda g_0$ on any nonvoid open part $\Gamma$ of $\partial\Omega$ however for all $g_0 \in H_{(1/2)}(\partial\Omega)$. In their proof they combined the theory which will be exposed in section 5.3 with use of Carleman type estimates (to "switch-off" $\partial\Omega \setminus \Gamma$).

To prove Theorem 5.2.2 when $c_j \in L_\infty(\Omega)$, we again make use of the relations (5.0.4) and obtain

$$\int_\Omega (c_2 - c_1)u_1 u_2 = 0 \tag{5.2.2}$$

for all solutions $u_1, u_2$ to equations (5.2.1) near $\overline{\Omega}$. Later on we will prove that $\operatorname{span}\{u_1 u_2\}$ is dense in $L_1(\Omega)$ (Corollary 5.3.5), which implies that $c_1 = c_2$.

So in the available theory, completeness of products of solutions of PDE plays a crucial role, which has been observed by many researchers, but the property of completeness for operators with variable coefficients was first established by Sylvester and Uhlmann in [SyU2]. We consider this property in Section 5.3 in more detail and generalize results of [SyU2].

Bukhgeim and Uhlmann [BuU] showed that one can use less boundary data in Theorems 5.2.1, 5.2.2. To formulate their result we fix a unit vector $e \in \mathbb{R}^n$ and define the following subsets of $\partial\Omega$: $\Gamma_{+,\varepsilon} = \partial\Omega \cap \{\varepsilon < \nu \cdot e\}$, $\Gamma_{-,\varepsilon} = \partial\Omega \cap \{\nu \cdot e < \varepsilon\}$. Let $\Lambda_{c_j,\Gamma,p}$ be the partial Dirichlet-to Neumann map mapping the Dirichlet data $g_0$ on $\partial\Omega$ into the Neumann data $\partial_\nu u$ on $\Gamma$ where $u$ solves the Schrödinger equation (5.2.1) in $\Omega$. Bukhgeim and Uhlmann proved

**Theorem 5.2.2′**. *Let* $n \geq 3$. *Let* $c_j \in L_\infty(\Omega)$. *Let* $0 < \varepsilon$.

*If equations* (5.2.1) *have the same partial Dirichlet-to-Neumann maps* $\Lambda_{c_j,\Gamma_{-,\varepsilon},p}$, *then* $c_1 = c_2$.

The proof repeats the proof of Theorem 5.2.2 if instead of Corollary 5.3.5 one uses Corollary 5.3.5′.

Now we give a stability result due to Alessandrini ([Al1], [Al3], Corollary 1.2).

**Theorem 5.2.3**. *Let* $n \geq 3$. *Let* $a_1, a_2$ *or* $c_1, c_2$ *satisfy the conditions*

$$\|a_j\|_{3,\infty}(\Omega) \leq M, \quad 1/E \leq a_j \leq E,$$
$$\|c_j\|_{1,\infty}(\Omega) \leq M. \tag{5.2.3}$$

*Then*

$$\|a_2 - a_1\|_2(\Omega) \leq \omega(\|\Lambda_2 - \Lambda_1\|)$$

*or*

$$\|c_2 - c_1\|_2(\Omega) \leq \omega(\|\Lambda_2 - \Lambda_1\|),$$

*where* $\omega(t) = CM|\ln t/C|^{-2/(n+1)}$ *and* $C$ *depends only on* $n, \Omega, E$.

PROOF. To prove Theorem 5.2.3 we use the substitution (5.2.1) and Theorem 5.1.5 and reduce the conductivity equation to the Schrödinger equation. We give the further argument for this equation. By using the identity (5.0.4), where $a_j = 1$, for the solutions $u_j$ constructed in Theorem 5.3.4 we obtain

$$\int_\Omega (c_1 - c_2)\exp(i\xi \cdot x)dx$$
$$= -\int_\Omega (c_1 - c_2)\exp(i\xi \cdot x)(w_1 + w_2 + w_1 w_2) + \int_{\partial\Omega} u_1(\Lambda_1 - \Lambda_2)u_2.$$

From Theorem 5.3.4 and the proof of Corollary 5.3.5 we have $\|w_j\|_2(\Omega) \leq CM/(|\xi| + R)$, where $R$ is a large number. From Lemma 5.3.1, from (5.3.3) and from Theorem 4.1 we have also that $\|u_j\|_{(1)}(\Omega) \leq CM\exp(R + |\xi|)$. So by trace theorems we can similarly bound $\|u_j\|_{(1/2)}(\partial\Omega)$. Letting $c$ be $(c_1 - c_2)\chi(\Omega)$ we obtain for its Fourier transformation

$$|\hat{c}|(\xi) \leq C((R + |\xi|)^{-1} + \exp(CR + C|\xi|)\|\Lambda_1 - \Lambda_2\|).$$

Letting $\rho = R + |\xi|$ and $\delta = \|\Lambda_1 - \Lambda_2\|$ we will minimize the right side with respect to $\rho > 0$. The minimum point $\rho_*$ exists and satisfies the equality

$$-\rho_*^{-2} + C\delta\exp(C\rho_*) = 0, \text{ or } \delta^{-1} = C\rho_*^2\exp(C\rho_*) \leq C\exp(C\rho_*),$$

so $\rho_* \geq -(\ln\delta + \ln C)/C$. This estimate and the relation for the minimum point imply that

$$|\hat{c}|(\xi) \leq C\rho_*^{-1}(1 + \rho_*^{-1}).$$

We can assume that $\delta < 1/C$ and drop the second term on the right side, obtaining

$$|\hat{c}|(\xi| \leq -C/(\ln\delta + \ln C)$$

and

$$(5.2.4) \qquad \int_{|\xi|<\rho_1} |\hat{c}(\xi)|^2 \leq C\delta_1^2\rho_1^{n-1}, \delta_1 = -1/(\ln\delta + \ln C).$$

This bound controls "lower frequencies" $\xi$, and to complete the proof we need to bound higher ones. The function $c$ is not smooth, but by using Exercise 5.1.6 and extension theorems we can replace it by a smooth function $c^*$ which is close to $c$.

Indeed, let $E$ be a standard extension operator from $H_{(1/2)}(\partial\Omega)$ into $H_{(1)}(\mathbb{R}^n)$. We can assume that $Eg$ is supported in some ball $B$. For $f \in H_{(1)}(\Omega)$ we let $g = f$ on $\partial\Omega$ and we define $f^*$ as $f$ on $\Omega$ and as $Eg$ on $\mathbb{R}^n \setminus \Omega$. We have

$$(5.2.5) \quad \|c^* - c\|_{(0)}(\mathbb{R}^n) = \|c^*\|_{(0)}(B \setminus \Omega) \leq C\delta_2, \;\; \delta_2 = \|c_2 - c_1\|_{(1/2)}(\partial\Omega)$$

by continuity of $E$. From Exercise 5.1.6 and interpolation inequalities for functions on $\partial\Omega$ (Theorem A4) we have $\delta_2 \leq CM\|\Lambda_2 - \Lambda_1\|^{1/6} = CM\delta^{1/6}$. On the other hand, since $c^* = c$ on $\Omega$,

$$\|c^*\|^2_{(1)}(\mathbb{R}^n) = \|c\|^2_{(1)}(\Omega) + \|c^*\|^2_{(1)}(B \setminus \overline{\Omega}) \leq CM.$$

Hence,

$$\int |\xi|^2|\hat{c}^*(\xi)|^2 d\xi \leq C^2M^2,$$

so the integral of $|\hat{c}^*|^2$ over $|\xi| > \rho_1$ is not greater than $M^2/\rho_1^2$. Using the Parseval identity

$$\int_{\rho_1<|\xi|} |\hat{c}(\xi)|^2 d\xi \leq 2\int |\hat{c}(\xi) - \hat{c}^*(\xi)|^2 d\xi + 2\int_{\rho_1<|\xi|} |\hat{c}^*(\xi)|^2 d\xi$$

$$2\|c - c^*\|^2_2(\mathbb{R}^n) + 2M^2/\rho_1^2.$$

Using again the Parseval identity and combining this inequality with (5.2.4) and (5.2.5) we yield

$$\|c\|^2_2(\Omega) = C\|\hat{c}\|^2_2 \leq C(\delta_1^2\rho_1^{n-1} + M^2/\rho_1^2 + \delta_2)$$

Again minimizing the right side with respect to $\rho_1$ we obtain for the minimum point $\rho_\bullet : (n-1)\delta_1^2\rho_\bullet^{n-2} - 2M^2\rho_\bullet^{-3} = 0$. Letting $\rho = \rho_\bullet$ and using that $\delta_1^2\rho_\bullet^{n-1} = 2M^2/((n-1)\rho_\bullet^2)$, we finally conclude that

$$\|c\|^2_2(\Omega) \leq CM^{2(n-1)/(n+1)}\delta_1^{4/(n+1)} + CM^2\delta^{1/3},$$

and by using the definition of $\delta_1$ in (5.2.4) the estimate for $c$ follows. □

Recently Mandache [Ma] showed that logarithmic type stability estimate is optimal in the sense that one cannot replace the logarithmic function by a power. Let $\Omega$ be the unit ball in $\mathbb{R}^n$, $2 \leq n$. Mandache combined properties of harmonic functions with an argument based on enthropy of functional spaces to prove that for any natural $m > 0$ and any there is a constant $C$ such that for any $\varepsilon$ one can find (real-valued) coefficients $c_1, c_2$ with the properties

$$\|\Lambda_2 - \Lambda_1\| \le exp(-\varepsilon^{-n/(M(2n-1))})\ \|c_2 - c_1\|_\infty(\Omega) = \varepsilon \ \text{ and } \ |c_j|_m < C,$$

where $\Lambda_j$ is the Dirichlet-to Neumann operator corresponding to the coefficient $c_j$ of the Scrödinger equation, $\|\Lambda\|$ is the norm of the operator $\Lambda$ from $L_2$ into $L_2$. For complex-valued $c$ he found an explicit following counterexample. Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be a function supported in $\{x \in \mathbb{R}^2 : |x| < 1/2, 1/4 < x_1\}$ with $\|\chi\|_\infty = 1$ and $c(x) = n^{-m}e^{n\phi}\chi(x)$. Here $\phi$ is the polar angle. It is shown in [Ma] that $|c|_m < C(m)$ and $\|\Lambda_c - \Lambda_0\| \le C(m)2^{-n/2}$ while obviously $\|c - 0\|_\infty = 1$. By using ideas of [Ma] Di Christo and Rondi replaced operator norms by more natural from $H_{(1/2)}(\partial\Omega)$ into $H_{(-1/2)}(\partial\Omega)$.

## 5.3 Completeness of products of solutions of PDE

The property of completeness of products for harmonic function in $\mathbb{R}^n$, $n \ge 2$, was observed by Calderon [C]. Apparently, it is not valid in $\mathbb{R}^1$ (linear combinations of linear functions are not dense). Since it plays a fundamental role in the thoery of inverse problems, we discuss its connection to potential theory (uniqueness theorems for Riesz potentials).

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \ge 2$. Let us consider functions $u_1, u_2$ that are harmonic near $\overline{\Omega}$. We will show that span$\{u_1u_2\}$ is dense in $L^1(\Omega)$, for $n \ge 3$. Assume that it is not so. Then by Hahn-Banach Theorem there is a nonzero measure $\mu$ supported in $\overline{\Omega}$ such that $\int u_1(y)u_2(y)d\mu(y) = 0$ for all such $u_1, u_2$. The functions $u_1(y) = |x - y|^{2-n} = u_2(y)$ are harmonic near $\overline{\Omega}$ when $x$ is outside $\overline{\Omega}$, so the Riesz potential $\int |x - y|^\alpha d\mu(y) = 0$ $(\alpha = -2)$ when $x$ is outside $\overline{\Omega}$. It was proven using asymptotic behavior of this potential at infinity by M. Riesz [Ri] or using the Fourier transformation by Isakov [Is4], p. 79, that when $\alpha \ne 2k$, $\alpha + n \ne 2k + 2$, the exterior Riesz potential determines a measure $\mu$ in a unique way, so we have $\mu = 0$, which is a contradiction.

Now we will demonstrate the Calderon's approach, which turned out to be fruitful for some equations with variable coefficients.

First we give some auxiliary results.

**Lemma 5.3.1**. *Let $\xi(0) \in \mathbb{R}^n\backslash\{0\}$, $n \ge 2$. Then there are $\zeta(1), \zeta(2) \in \mathbb{C}^n$ such that*

(5.3.1) $$\zeta(1) \cdot \zeta(1) = 0 = \zeta(2) \cdot \zeta(2), \qquad \zeta(1) + \zeta(2) = \xi(0).$$

*If $n \ge 3$, then for any $R > 0$ there are such $\zeta(1), \zeta(2)$ with the additional property*

(5.3.2) $$R + |\xi(0)|/2 \le |\zeta(j)|, \quad |\Im\zeta(j)| \le |\xi(0)|/2 + R.$$

PROOF. Let $\zeta(1) = \xi + i\eta$. Then $\zeta(2) = \xi(0) - \xi - i\eta$, where $\xi, \eta$ are vectors in $\mathbb{R}^n$ to be found. The relations (5.3.1) are equivalent to the equalities $\xi \cdot \xi = \eta \cdot \eta$, $\xi \cdot \eta = 0$, $|\xi(0)|^2 = 2\xi \cdot \xi(0)$, $\xi(0) \cdot \eta = 0$.

If $n = 2$, let $\xi(0) = te_1$, and $e_1, e_2$ be an orthonormal basis in $\mathbb{R}^2$. When the $\xi_j$ are coordinates of $\xi$ in this basis then all solutions to the above system of the equations for $\xi, \eta$ are $\xi_1 = t/2, \xi_2 = 0, \eta_1 = 0, \eta_2 = t/2$ or $-t/2$.

If $n \geq 3$, then we similarly have the solutions

$$\xi_1 = t/2, \xi_2 = 0, \xi_3 = R, \eta_1 = 0, \eta_2 = (t^2/4 + R^2)^{1/2}, \eta_3 = 0.$$

Observing that $|\Im\zeta(j)| = |\eta| = (|\xi(0)|^2/4 + R^2)^{1/2}$ and using the inequality $a^2 + b^2 \leq (a+b)^2$ for nonnegative $a, b$, we complete the proof. □

Now we are ready to demonstrate Calderon's proof of completeness of products of harmonic functions.

Since linear combinations of exponential functions $\exp(i\xi(0)\cdot x)$, $\xi(0) \in \mathbb{R}^n\backslash\{0\}$ are dense (in $L_2(\Omega)$), it suffices to find two harmonic functions whose product is this exponential function. We let $u_1(x) = \exp(i\zeta(1)\cdot x)$, $u_2(x) = \exp(i\zeta(2)\cdot x)$, where the $\zeta(j)$ satisfy conditions (5.3.1), which guarantee that $u_1, u_2$ are harmonic. Such $\zeta(j)$ exist by Lemma 5.3.1. By multiplying the exponential functions and using condition (5.3.1) again, we obtain $\exp(i\xi(0)\cdot x)$.

This approach was transferred by Sylvester and Uhlmann [SyU2] onto solutions of the multidimensional Schrödinger equation (5.2.1). Motivated by the Calderon's idea and by geometrical optics, they suggested using almost exponential solutions to this equation,

$$u_j(x) = \exp(i\zeta(j)\cdot x)(1 + w_j(x)), \tag{5.3.3}$$

with the property that $w_j$ goes to zero (in $L_2(\Omega)$) when $R$ (from Lemma 5.3.1) goes to infinity. We will extend their method to the more general equations

$$(P_j(-i\partial) + c_j)u_j = 0 \tag{5.3.4}$$

(which include parabolic and hyperbolic equations with constant leading coefficients) by using the following auxiliary result.

**Lemma 5.3.2**. *Let $P$ be a linear partial differential operator of order $m$ in $\mathbb{R}^n$ with constant coefficients. Then there is a bounded linear operator $E$ from $L_2(\Omega)$ into itself such that*

$$P(-i\partial)Ef = f \text{ for all } f \in L_2(\Omega) \tag{5.3.5}$$

*and*

$$\|QEf\|_2(\Omega) \leq C \sup(\tilde{Q}(\xi)/\tilde{P}(\xi))\|f\|_2(\Omega), \tag{5.3.6}$$

*where $C$ depends only on $m$, $n$,* diam $\Omega$, *and* sup *is over* $\xi \in \mathbb{R}^n$.

Here $\tilde{P}(\xi) = \left(\sum_{|\alpha|\leq m} |\partial_\xi^\alpha P(\xi)|^2\right)^{1/2}$.

Actually, this result was obtained by Ehrenpreis and Malgrange in the 1950s. For complete proofs we refer to the book of Hörmander [Hö2] and to the paper of Isakov [Is7].

**Theorem 5.3.3**. *Let $\Sigma_0$ be an open nonvoid subset of $\mathbb{R}^n$. Suppose that for any $\xi(0) \in \Sigma_0$ and for any number $R$ there are solutions $\zeta(j)$ to the algebraic equations*

$$\zeta(1) + \zeta(2) = \xi(0), \quad P_j(\zeta(j)) = 0 \text{ with } |\zeta(j)| > R. \tag{5.3.7}$$

*and there is a positive number $C$ such that for these $\zeta(j)$ we have*

$$|\zeta(j)| \leq C\tilde{P}_j(\xi + \zeta(j)) \text{ for all } \xi \in \mathbb{R}^n. \tag{5.3.8}$$

*If $f \in L_\infty(\Omega)$ and*

$$\int_\Omega f u_1 u_2 = 0 \tag{5.3.9}$$

*for all $L_2$-solutions $u_j$ to the equations* (5.3.4) *near $\overline{\Omega}$, then $f = 0$.*

This theorem claims that linear combinations of products of solutions of these equations are dense in $L_1(\Omega)$. It easily follows from the following construction of almost exponential solutions to the equations (5.3.4).

**Theorem 5.3.4**. *Suppose that conditions* (5.3.7), (5.3.8) *are satisfied.*

*Then for any $\xi(0) \in \Sigma_0$ there are solutions to equations* (5.3.4) *near $\overline{\Omega}$ of the form* (5.3.3)*, where $\|w_j\|_2(\Omega) \leq C/|\zeta(j)|$ and $C$ depends only on $\|c_j\|_\infty(\Omega)$ and on* diam $\Omega$.

PROOF. From Leibniz's formula we have

$$P(-i\partial)(uv) = \sum_\alpha P^{(\alpha)}(-i\partial)u(-i\partial)^\alpha v/\alpha!.$$

Letting $u = \exp(i\zeta \cdot x)$, $v = 1 + w_j$ and using the relations $P^{(\alpha)}(-i\partial)u = P^{(\alpha)}(\zeta)\exp(i\zeta \cdot x)$ as well as the equality $P_j(\zeta(j)) = 0$, we conclude that $u_j$ in (5.3.3) solves equation (5.3.4) if and only if

$$P_j(-i\partial + \zeta(j))w_j = \sum_\alpha P_j^{(\alpha)}(\zeta(j))(-i\partial)^\alpha w_j/\alpha! = -c_j(1 + w_j). \tag{5.3.10}$$

Let $E$ be the operator of Lemma 5.3.2 for $P(-i\partial) = P_j(-i\partial + \zeta(j))$. Then any solution $w_j$ to the equation

$$w_j = -E(c_j(1 + w_j)) \quad \text{in } L_2(\Omega) \tag{5.3.11}$$

is a solution to (5.3.4). From condition (5.3.7) and from the estimate (5.3.6) with $Q(\partial) = |\zeta(j)|$ we have $\|Ef\|_2(\Omega) \leq C|\zeta(j)|^{-1}\|f\|_2(\Omega)$. So using conditions (5.3.7), (5.3.8) we can choose $|\zeta(j)|$ so large that the operator from the right side of equation (5.3.11) is a contraction of the ball $\{\|w\|_2(\Omega) \leq 2C\|c_j\|_\infty(\Omega)$ $(meas_n\,\Omega)^{1/2}/|\zeta(j)|\}$ in $L_2(\Omega)$ into itself. By the Banach contraction theorem there is a solution $w_j$ to the equation (5.3.11) in this ball. This proves Theorem 5.3.4. □

PROOF OF THEOREM 5.3.3. Let $\xi(0) \in \Sigma_0$ and let $u_1, u_2$ be solutions constructed in Theorem 5.3.4. Then $\zeta(1) + \zeta(2) = \xi(0)$, and functions $u_1u_2(x) = \exp(i\xi(0) \cdot x)(1 + w_1 + w_2 + w_1w_2)$ converge to $\exp(i\xi(0) \cdot x)$ in $L_1(\Omega)$ as the $|\zeta(j)|$ go to infinity. So the Fourier transform of $f$ (extended as zero outside $\Omega$) is zero on $\Sigma_0$. Since $f$ is compactly supported, its Fourier transform is analytic on $\mathbb{R}^n$. Therefore it is zero everywhere, and so is $f$. This completes the proof. □

**Corollary 5.3.5**. *Let $n \geq 3$. If (5.3.9) is valid for all solutions $u_j$ to the equations $-\Delta u_j + c_j u_j = 0$, $j = 1, 2$, near $\overline{\Omega}$, then $f = 0$.*

PROOF. To prove this result we check the conditions of Theorem 5.3.3 for $P_j(\zeta) = \zeta_1^2 + \cdots + \zeta_n^2$. Let $\xi(0) \in \mathbb{R}^n$. Due to rotational invariancy we may assume $\xi(0) = (\xi_1(0), 0, \ldots, 0)$. The vectors

$$\zeta(1) = (\xi_1(0)/2, i(\xi_1^2(0)/4 + R^2)^{1/2}, R, 0, \ldots, 0),$$
$$\zeta(2) = (\xi_1(0)/2, -i(\xi_1^2(0)/4 + R^2)^{1/2}, -R, 0, \ldots, 0)$$

are solutions to the equation $\zeta \cdot \zeta = 0$ with the absolute values greater than $R$, so condition (5.3.7) is satisfied. To check condition (5.3.8) we observe that

$$\begin{aligned}\tilde{P}^2(\xi + \zeta) &\geq |2\zeta_1 + 2\xi_1|^2 + \cdots + |2\zeta_n + 2\xi_n|^2 + 12 \\ &\geq 4\left(|\Im\zeta_1|^2 + \cdots + |\Im\zeta_n|^2\right) \geq |\zeta|^2,\end{aligned}$$

provided that $\zeta \cdot \zeta = 0$, because then $|\Re\zeta| = |\Im\zeta|$. This completes the proof. □

In the paper [SyU2] the authors used the more standard fundamental solution $E$ (the Faddeev Green's function) of the operator $\Delta + 2i\zeta \cdot \nabla$, and they obtained estimates (5.3.6) in the Sobolev spaces $H_\delta^m(\mathbb{R}^n)$, which are constructed from the weighted $L_{2;\delta}(\mathbb{R}^n)$-spaces with the norm $\|f(x)(1 + |x|^2)^{\delta/2}\|_2(\mathbb{R}^n)$. They showed also that $E$ is invertible in these weighted spaces. Hähner [Ha] observed that in the Sylvester-Uhlmann scheme it suffices to use periodic solutions of the Schrödinger equation, because $c$ is compactly supported. To explain his idea we let $\Omega_R = (-R, R) \times \cdots \times (-R, R)$ be the cube in $\mathbf{R}^n$ and we consider functions $f(x)$ which are $2R$-periodic with respect to all variables $x_1, \ldots, x_n$. Hähner proved that there is a periodic fundamental solution $E_p$ of the operator $\Delta + 2i\zeta \cdot \nabla$ satisfying the bound

$$\|E_p f\|_2(\Omega_R) \leq R/(\pi|\Im\zeta|)\|f\|_2(\Omega_R).$$

The bound on the norm of this operator is explicit and one can use it instead of $E$ in the proof of Theorem 5.3.4 obtaining explicit bounds on constants $C$. A proof of this estimate is straightforward if one uses Fourier series representations for $f$.

**Exercise 5.3.6**. Let $P = -\Delta - 2\zeta \cdot \nabla$, $\zeta \cdot \zeta = 0$, and $E$ be its regular fundamental solution. Prove the estimate $\|Ew\|_{(1)}(\Omega) \leq C\|w\|_{(0)}(\Omega)$ for all functions $w \in L_2(\Omega)$ that are zero outside $\Omega$ with $C$ depending only on $\operatorname{diam}\Omega$ and the dimension $n$ of the space. Using this result prove that there are solutions $u_j$ to the Schrödinger equation $(-\Delta + c_j)u_j = 0$ f the form (5.3.3) with

$$|\zeta(j)|\|w_j\|_2(\Omega) + \|w_j\|_{(1)}(\Omega) \leq C.$$

{*Hint*: make use of Lemma 5.3.2 and of the proof of Corollary 5.3.5.}

**Exercise 5.3.7**. Let $P = -\Delta - 2\zeta \cdot \nabla$, $\zeta \cdot \zeta = \lambda^2$, $\lambda \in \mathbb{R}_+$, and $E$ be its regular fundamental solution. Prove the estimate $\|Ew\|_{(0)}(\Omega) \leq C/\lambda \|w\|_{(0)}(\Omega)$ for all $w$ from Exercise 5.3.6 with $C$ depending on the same parameters as in that exercise.

Now we will demonstrate an approach of Bukhgeim and Uhlmann [BuU] based on Carleman estimates which leads to uniqueness of the coefficient $c$ (and hence of scalar principal coefficient $a$) from some partial boundary data. We will use notation $\Gamma_+$, $\Gamma_-$ from section 5.2.

**Lemma 5.3.8**. *Let $e$ be a unit vector in $\mathbb{R}^n$ and $\varphi(x) = -e \cdot x$.*

*Then there is a constant $C$ depending only on $\Omega$, $\|c\|_\infty(\Omega)$ such that*

$$\tau^2/C \int_\Omega |u|^2 e^{2\tau\varphi} + 2\tau \int_{\Gamma_+} e \cdot \nu |\partial_\nu|^2 e^{2\tau\varphi}$$

$$\leq \int_\Omega |-\Delta u + cu|^2 e^{2\tau\varphi} - 2\tau \int_{\Gamma_-} e \cdot \nu |\partial_\nu u|^2 e^{2\tau\varphi}$$

*for all functions $u \in H_{(2)}(\Omega)$ with $u = 0$ on $\partial\Omega$.*

A proof can be obtained following advices to solve Exercise 3.4.5. Lemma 5.3.8 and the standard scheme from functional analysis (similar to proofs of Lax-Milgram theorem) lead to the following

**Corollary 5.3.9**. *There is an operator $E_\Omega$ from $L_2(\Omega)$ into $H_{(2)}(\Omega)$ and a constant $C$ depending only on $\Omega$ and $\|c\|_\infty(\Omega)$ such that $v = E_\Omega f$ solves the boundary value problem*

$$-\Delta v + cv = f \text{ in } \Omega, \ v = 0 \text{ on } \Gamma_-$$

*and*

$$\int_\Omega |v|^2 e^{2\tau\varphi} \leq C/\tau^2 \int_\Omega |f|^2 e^{2\tau\varphi}.$$

Corollary 5.3.9 is an analogue of Lemma 5.3.2 for a bounded domain and $E_\Omega$ can be viewed as a special fundamental solution satisfying a partial boundary condition. Using this corollary in the proof of Theorem 5.3.3 instead of Lemma 5.3.2 one can construct almost exponential solutions (5.3.3) to the Schrödinger equation in $\Omega$ which satisfy partial boundary conditions.

Now we give an outline of a proof of Theorem 5.2.2′. Let $\xi \in \mathbb{R}^n$ be orthogonal to $e$, $|\xi| < \tau$, and $\eta \in \mathbb{R}^n$ be orthogonal to both $e$ and $\xi$ with $|\eta|^2 = \tau^2 - |\xi|^2$. Let us define $\zeta(1) = -\tau e - i(\xi + \eta)$, $\zeta(2) = \tau e - i(\xi - \eta)$. By Theorem 5.3.4 there is an almost exponential solution $u_1^*(x) = e^{\zeta(1)\cdot x}(1 + w_1^*(x))$ with $\|w_1^*\|_2(\Omega) \leq C/\tau$. Using Corollary 5.3.9 instead of theorem 5.3.3 and repeating the proof of Theorem 5.3.4 one can construct a solution $u_2(x) = e^{\zeta(2)\cdot x}(1 + w_2(x))$ with $w_2 = 0$ on $\Gamma_-$ and $\|w_2\|_2(\Omega) \leq C/\tau$ with $\|w_2\|_2(\Omega) \leq C/\tau$. Let $u_1$ solve the Dirichlet problem

$-\Delta u_1 + c_1 u_1 = 0$ in $\Omega$ with the Dirichlet data $u_1 = u_2$ on $\partial\Omega$. We let $u = u_2 - u_1$. Then

$$-\Delta u + c_1 u = f u_2 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

and since the partial Dirichlet-to Neumann maps for both equations coincide we have $\partial_\nu u = 0$ on $\Gamma_-$. From the Green's formula

$$\int_\Omega f u_2 u_1^* = \int_{\Gamma_{+,\varepsilon}} \partial_\nu u u_1^*. \tag{5.3.12}$$

The crucial step of the proof is to show that for our particular choice of $u_2, u_1^*, u$ the right side goes to zero as $\tau$ goes to infinity. Suppose it is true, then as in the proof of Theorem 5.3.3 the left side converges to the Fourier transformation $\hat{f}(\xi)$ and hence $\hat{f}(\xi) = 0$ for all $\xi$ which are orthogonal to $e$. Since $0 < \varepsilon$ the conditions of Theorem 5.2.2' are satisfied for small perturbations of $e$ and hence $\hat{f}(\xi) = 0$ for $\xi$ in an open (conical) subset of $\mathbb{R}^n$. The Fourier transformation is analytic because $f$ is compactly supported, so it is zero for all $\xi$ and $f = 0$ by applying the inverse Fourier transformation.

To complete the proof it suffices to show that the right side in (5.3.12) goes to zero as $\tau$ goes to infinity. Using the form of $u_1^*$ and the Hölder inequality we yield

$$\left|\int_{\Gamma_{+,\varepsilon}} \partial_\nu u u_1^*\right| \le \left(\int_{\Gamma_{+,\varepsilon}} |1 + w_1^*|^2\right)^{1/2} \left(\int_{\Gamma_{+,\varepsilon}} |\partial_\nu u|^2 e^{2\tau\varphi}\right)^{1/2}$$

By Exercise 5.3.6, we have $\|w_1^*\|_{(1)}(\Omega) \le C$ and hence by trace theorems the boundary integral of $|1 + w_1^*|^2$ is bounded (with respect to $\tau$). The second boundary integral is bounded by $C/\tau$ due to Lemma 5.3.8, because $-\Delta u + c_2 u = f u_1$ in $\Omega$ and $u = 0$ on $\partial\Omega$, $\partial_\nu u = 0$ on $\Gamma_-$.

## 5.4 Recovery of several coefficients

First, we observe that generally one cannot expect to find all coefficients of the second-order elliptic equation

$$\operatorname{div}(-a\nabla u) + b \cdot \nabla u + cu = 0 \qquad \text{in } \Omega \tag{5.4.1}$$

from its Dirichlet-to-Neumann map due to the invariancy of this map with respect to multiplication of $u$ by a smooth positive function that is 1 near $\partial\Omega$. Even when $b = 0$ one can transform (5.4.1) into the equation $-\Delta v + c^* v = 0$ by using the substitution $u = a^{-1/2} v$ as in (5.2.1) without changing $\Lambda$. One way to assure uniqueness is to use several frequencies in the scattering situation, which will be considered below. However, some uniqueness results can be obtained.

In the following result we consider two equations (5.4.1) with $a = 1$, $b = b_j \in H_{2,\infty}(\Omega)$, and $c = c_j \in L_\infty(\Omega)$, $j = 1, 2$.

**Theorem 5.4.1**. *Assume that $\Lambda_1 = \Lambda_2$ and $b_j, c_j$ are compactly supported in a simply connected $\Omega$. Let $(n = 3)$.*

*There exists a positive $\varepsilon(\Omega)$ such that if $\|b_j\|_\infty(\Omega) < \varepsilon$, then one has*

$$curl\, b_1 = curl\, b_2 \text{ and } 4c_1 + b_1 \cdot b_1 - 2\,\mathrm{div}\, b_1 = 4c_2 + b_2 \cdot b_2 - 2\,\mathrm{div}\, b_2 \text{ on } \Omega. \tag{5.4.2}$$

*If $b_j, c_j \in C^\infty(\overline{\Omega})$ and $\Re b_j = 0$, $\Im c_j = 0$, then* (5.4.2) *holds without the smallness assumption*

Proofs are available from the papers of Sun [Su1], and of Nakamura, Sun, and Uhlmann [NaSU] (global uniqueness of smooth $b$, $c$ in the three-dimensional case). Local uniqueness in the three-dimensional case was also obtained by Sun [Su1] under the additional assumption $\Re b_j = 0$, $\Im c_j = 0$. This type of conditions guarantees that the elliptic operator is self adjoint. Eskin and Ralston and in general case of systems Eskin [Es1] obtained global uniqueness for the corresponding scattering problem, which implies (5.4.2) under the self-adjointness assumption and under the regularity assumptions $b \in C^6(\overline{\Omega})$, $c \in C^5(\overline{\Omega})$ when $\Omega$ is a domain in $\mathbb{R}^3$. We give a proof of the local uniqueness result for $b_j$, $c_j$ (assuming $\|b_j\|_\infty(\Omega) < \varepsilon$ and not assuming self-adjointness) in the three-dimensional case.

The substitution $u = e^\phi v$ transforms equation (5.4.1) for $u$ into the equation

$$-\Delta v + (b - 2\nabla\phi) \cdot \nabla v + (c + b \cdot \nabla\phi - \Delta\phi - \nabla\phi \cdot \nabla\phi)v = 0.$$

When $\phi = 0$, $\partial_\nu \phi = 0$ on $\partial\Omega$, the Dirichlet-to-Neumann operator for the new equation will be the same as for the old one and we cannot expect complete recovery of $b$.

PROOF OF THEOREM 5.4.1, local part.

We will make use of almost exponential solutions

$$u_1(x) = \exp(i\zeta(1) \cdot x + \phi_1)(1 + w_1),\; u_2^*(x) = \exp(i\zeta(2) \cdot x + \phi_2)(1 + w_2) \tag{5.4.3}$$

of the equations

$$-\Delta u_1 + b_1 \cdot \nabla u_1 + c_1 u_1 = 0, \quad -\Delta u_2^* - b_2 \cdot \nabla u_2^* + (c - \mathrm{div}\, b_2)u_2^* = 0.$$

As in the proof of Theorem 5.3.4, the equation for $u_1$ is equivalent to the equation

$$\begin{aligned} &-\Delta w_1 - 2i\zeta(1) \cdot \nabla w_1 \\ &\quad = (\nabla\phi_1 - b_1) \cdot \nabla w_1 + (2i\zeta(1) \cdot \nabla\phi_1 + |\nabla\phi_1|^2 \\ &\qquad + \Delta\phi_1 - i\zeta(1) \cdot b_1 - c_1)(1 + w_1). \end{aligned}$$

This equation will follow from the equality

$$w_1 = -E(b_{\bullet 1} \cdot \nabla w_1 + c_{\bullet 1}(1 + w_1)) \text{ in } L_2(\Omega), \tag{5.4.4}$$

where $b_{\bullet 1} = \nabla\phi_1 - b_1$, $c_{\bullet 1} = -c_1 + |\nabla\phi_1|^2 + \Delta\phi_1 + \cdots$. Here $\|b_{\bullet_1}\|_\infty(\Omega) \le C\|b_1\|_\infty(\Omega)$, $\|c_{\bullet_1}\|_\infty(\Omega) \le C$ (independent of $\zeta(1)$), provided that $\phi_1 = \phi_1(x, \zeta(1))$ satisfies the "transport equation"

$$2\zeta(1) \cdot \nabla\phi_1 = \zeta(1) \cdot b_1 + \cdots, \tag{5.4.5}$$

where ... denotes terms uniformly bounded (with respect to $\zeta(1)$), and in addition,

$$\|\phi_1\|_\infty(\Omega) \le C\|b_1\|_\infty(\Omega),\ \|\phi_1\|_{2,\infty}(\Omega) \le C.$$

We have a similar equation for $w_2$.

Referring to the proof of Corollary 5.3.5 and to Exercise 5.3.6, we claim that the operator $E$ from $L_2(\Omega)$ into $L_2(\Omega)$ and into $H_{(1)}(\Omega)$ has operator norms bounded by $C/|\zeta(j)|$ and by $C$.

We claim that the operator from the right side of (5.4.4) maps the convex compact set in $L_2(\Omega)$

$$W_1 = \{w \in L_2(\Omega) : \|w\|_2(\Omega) \le C_1/|\zeta(1)|,\ \|w\|_{(1)}(\Omega) \le C_1\},\ C_1 \\ = 2C^2 \operatorname{vol}\Omega^{1/2},$$

into itself, provided that

$$\|b_1\|_\infty(\Omega) < 1/(4C^2), \quad |\zeta(1)| > 4C^2. \tag{5.4.6}$$

Indeed,

$$\begin{aligned}
&\|E(b_{\bullet_1}\cdot\nabla w + c_{\bullet_1}(1+w))\|_2(\Omega)\\
&\quad \le C|\zeta(1)|^{-1}(\|b_{\bullet_1}\|_\infty(\Omega)\|w\|_{(1)}(\Omega) + \|c_{\bullet_1}\|_\infty(\Omega)(\operatorname{vol}\Omega^{1/2} + \|w\|_2(\Omega)))\\
&\quad \le C|\zeta(1)|^{-1}(\|b_{\bullet_1}\|_\infty(\Omega)C_1 + \|c_{\bullet_1}\|_\infty(\Omega)(\operatorname{vol}\Omega^{1/2} + C_1|\zeta(1)|^{-1}))\\
&\quad \le (C_1/4 + C^2\operatorname{vol}\Omega^{1/2} + C_1C^2|\zeta(1)|^{-1})|\zeta(1)|^{-1} \le C_1/|\zeta(1)|,
\end{aligned}$$

due to conditions (5.4.6). Similarly, one can check that

$$\|E(b_{\bullet_1}\cdot\nabla w + c_{\bullet_1}(1+w))\|_{(1)}(\Omega) \le C_1.$$

By the Schauder-Tikhonov theorem there is a solution $w_1$ to equation (5.4.4) in $W_1$. We treat $w_2$ in the same way. So we have the exponential solutions (5.4.3), where

$$\|w_j\|_2(\Omega) \le C/|\zeta(j)|,\ \|w_j\|_{(1)}(\Omega) \le C. \tag{5.4.7}$$

Since $\Lambda_1 = \Lambda_2$, the relation (5.0.3) gives

$$\begin{aligned}
0 &= \int_\Omega ((b_2-b_1)\cdot\nabla u_1u_2^* + (c_2-c_1)u_1u_2^*)\\
&= \int_\Omega ((b_2-b_1)\cdot(i\zeta(1)+\nabla\phi_1)(1+w_1) + \nabla w_1)(1+w_2)\\
&\quad + (c_2-c_1)(1+w_1)(1+w_2))\exp(i\xi(0)\cdot x + \phi_1 + \phi_2)dx,
\end{aligned} \tag{5.4.8}$$

where $\zeta(1) = (|\xi(0)|/2, i(R^2+|\xi(0)|^2/4)^{1/2}, R)$ (see the proof of Corollary 5.3.5). Here $R$ is any real number, and the orthonormal coordinate system is chosen in such a way that its first basis vector is parallel to $\xi(0)$. Observe that for large $R$ we have $\zeta(1) = (0, \quad i|R|, R) + \cdots$ and $\zeta(2) = -\zeta(1) + \cdots$, where ... denotes terms bounded with respect to $R$. We have the condition (5.4.5) for $\phi_1$ and a similar condition for $\phi_2$,

$$2\zeta(2)\cdot\nabla\phi_2 = -\zeta(2)\cdot b_2 + \cdots. \tag{5.4.9}$$

We claim that there are functions $\phi_1, \phi_2$ satisfying conditions (5.4.5), (5.4.9) with the bounds $\|\phi_j\|_{2,\infty}(\Omega) \leq C$. Indeed, we can take as $\phi_1$ the solution to the Cauchy-Riemann equations

$$2i\partial_2\phi_1 + 2\partial_3\phi_1 = ib_{12} + b_{13}$$

given by the Cauchy integral in the plane $\{e_2, e_3\}$. Then the bounds on $\phi_1$ follow from the regularity assumptions about $b_j$. Here $b_{1k}$ is the $k$th component of the vector $b_1$ in the basis $e_1, e_2, e_3$. We construct $\phi_2$ similarly. Observe that the $\phi_j$ do not depend on $R$. Adding (5.4.9) to (5.4.5), we derive that $2\zeta(1) \cdot \nabla(\phi_1 + \phi_1) = \zeta(1) \cdot (b_1 - b_2) + \ldots$. Dividing both parts by $R$, setting $\phi = \phi_1 + \phi_2$, $b = b_1 - b_2$, and letting $R \to +\infty$, we obtain

$$2i\partial_2\phi + 2\partial_3\phi = ib_2 + b_3. \tag{5.4.10}$$

Using, in addition, the bounds (5.4.7) on $w_1, w_2$, dividing (5.4.8) by $R$, and letting $R \to +\infty$, we conclude that

$$\int_\Omega ((\quad ib_2 + b_3)\exp(\phi))\exp(i\xi(0)\cdot x)dx = 0.$$

Taking $\xi(0) = te_1$, and using that the factor of the integrand in the parentheses does not depend on $t$, by uniqueness of the Fourier transformation in $x_1$ we obtain that

$$\int_{\{x_1=t\}} (\quad ib_2 + b_3)\exp(\phi)\,dx = 0$$

for all $t$. According to equation (5.4.10) the integrand is $2(\quad i\partial_2 + \partial_3)\exp(\phi)$, so we can integrate by parts and obtain the equality

$$0 = \int_{\{x_1=t\}\cap\partial B} \exp(\phi)(\quad i\nu_2 + \nu_3)d\sigma = i\int_{\{x_1=t\}\cap\partial B} \exp(\phi)(-dx_2 + idx_3) \tag{5.4.11}$$

for any ball $B$ containing $\overline{\Omega}$. Since $b = 0$ outside $\Omega$, from equation (5.4.10) we conclude that $\phi$ (and therefore $\exp(\phi)$) is an analytic function of $z = x_3 + ix_2$. By the construction, $\phi$ goes to zero as $z \to \infty$, so $\phi(z) = az^{-1} + O(z^{-2})$. From (5.4.11) it follows that the residue of $\exp(\phi)$ at $\infty$ is zero, so $a = 0$. Since $\phi$ is the Cauchy potential of $ib_2 + b_3$, it is equivalent to the relation

$$\int_{\{x_1=t\}} (\quad ib_2 + b_3)\,dx_2\,dx_3 = 0.$$

Repeating the same argument when $R \to -\infty$, we obtain the same relation with $-ib_2$ instead of $ib_2$. So the integral of any tangential component $\tau b$ of $b$ (with respect to the plane $\{x \cdot e_1 = t\}$) is zero:

$$\int_{\{x\cdot e_1=t\}} \tau b = 0.$$

The last step of the proof of uniqueness of $b_j$ is to show that the last equality implies that

$$\int_{\{x\cdot e_1<t\}} \operatorname{curl} b = 0. \tag{5.4.12}$$

Indeed, write all vectors and integrals in the coordinates $x_1, x_2, x_3$ corresponding to the orthonormal basis $e_1, e_2, e_3$. Then $b = (b_1, b_2, b_3)$,

$$\operatorname{curl} b = (\partial_2 b_3 - \partial_3 b_2, \partial_1 b_3 - \partial_3 b_1, \partial_1 b_2 - \partial_2 b_1),$$

and the integral (5.4.12) is over the half space $\{x_1 < t\}$. Since $b$ is compactly supported, we conclude that the integrals of the partial derivatives in the tangential directions $\partial_2, \partial_3$ are zero, and the integral (5.4.12) is equal to

$$\int_{\{x_1<t\}} (0, \partial_1 b_3, \partial_1 b_2) = \int_{\{x_1=t\}} (0, b_3, b_2) dS$$

according to the integration by parts formula. The components $b_2, b_3$ are tangential components of $b$, so as observed in the preceding item, the last integral is zero. Finally, we have the equality (5.4.12) for any unit vector $e_1$ and any real number $t$. Differentiating (5.4.12) with respect to $t$, we obtain that the integrals of $\operatorname{curl} b$ over all planes $\{x \cdot e_1 = t\}$ are zero. In other words, the function $\operatorname{curl} b$ has zero Radon transform. The known results (Corollary 7.1.3) imply then that this function is zero.

We have proved that $\operatorname{curl} b_1 = \operatorname{curl} b_2$.

Since $\Omega$ is simply connected and $\operatorname{curl} b = 0$, there is a solution $\phi$ to the equation $2\nabla\phi = -b = b_2 - b_1$ in $\Omega$. We can assume that $\phi$ is 0 at a boundary point. Then $\phi = 0$ on $\partial\Omega$ due to the condition $b = 0$ on $\partial\Omega$ and connectedness of $\partial\Omega$. As observed in Section 5.4 earlier, the substitution $u_2 = e^{\phi} v_2$ transforms the equation for $u_2$ into the equation

$$-\Delta v_2 + (b_2 - 2\nabla\phi) \cdot \nabla v_2 + (c_2 + b_2 \cdot \nabla\phi - \Delta\phi - \nabla\phi \cdot \nabla\phi)\, v_2 = 0$$

with the coefficient $b_2 - 2\nabla\phi = b_1$ due to the choice of $\phi$. Since $\phi = 0$ and $\nabla\phi = 0$ on $\partial\Omega$, this equation and the equation for $u_2$ have the same Dirichlet-to-Neumann maps, and so do the equations for $u_1$ and $v_2$. Let $c_{\bullet_2}$ be the coefficient of $v_2$ of the transformed equation. Arguing as above, we obtain the orthogonality relation (5.4.8) with $v_2^*$ instead of $u_2^*$ and without the terms with $b_j$, so

$$\int_\Omega (c_1 - c_{\bullet_2})(1 + w_1)(1 + w_2) \exp(i\xi(0) \cdot x + \phi_1 + \phi_2) = 0, \tag{5.4.13}$$

where the $w_j$ satisfy the estimates (5.4.7) and the phase functions $\phi_1, \phi_2$ satisfy the relations $2\zeta(1) \cdot \nabla\phi_1 = \zeta(1) \cdot b_1 + \cdots$ and $2\zeta(2) \cdot \nabla\phi_2 = -\zeta(2) \cdot b_1 + \cdots$. We recall that as above, ... denotes the terms bounded with respect to $R$, and $\zeta(1) = -\zeta(2) + \cdots$. Choosing $\phi_1, \phi_2$ as before, we get $\phi_1 = -\phi_2$, so we have the relation (5.4.13) without $\phi_j$. Letting $R$ go to $+\infty$, we conclude that the Fourier transform of $c_1 - c_{\bullet_2}$ is zero, so $c_1 = c_{\bullet_2}$.

The proof is complete. □

Now we consider an important and interesting case of anisotropic conductivity, i.e., equation (5.4.1) where $a$ is an $n \times n$-matrix $(a^{jk})$, $b = 0$, $c = 0$. It has been observed by Luc Tartar (see the description in the paper of Kohn and Vogelius [KoV3]) that if $\Psi \in C^1(\overline{\Omega})$ is a diffeomorphism of $\Omega$ onto itself that is the identity on $\partial\Omega$, then the new matrix

$$a^{\bullet}(x) = |\det \Psi'|^{-1\,t}\Psi' a \Psi'(\Psi^{-1}(x)), \tag{5.4.14}$$

where $\Psi'$ is the differential of $\Psi$, produces the same Dirichlet-to-Neumann map. To prove this we change variables $x = x(y) = \Psi^{-1}(y)$ in the definition of a generalized solution to the equation $\operatorname{div}(a\nabla u) = 0$:

$$\begin{aligned} 0 &= \int_\Omega a(x)\,\nabla u(x)\cdot\nabla\phi(x)dx \\ &= \int_\Omega a^*(y)\Psi'(y)\nabla_y u^*\cdot\Psi'(y)\nabla_y\phi^*|\det\Psi'|^{-1}dy \\ &= \int_\Omega |\det\Psi'|^{-1t}\Psi' a^*\Psi'\nabla_y u^*\cdot\nabla_y\phi^* dy, \quad \phi\in C_0^1(\Omega), \end{aligned}$$

where $a^*(y) = a(\Psi^{-1}(y))$. So $u^*(y)$ is a solution to the anisotropic conductivity equation with the conductivity matrix given by (5.4.14). If the diffeomorphism $\Psi$ is identical at the boundary and its differential is the identity matrix at the boundary, then $a$ and $a^*$ produce the same Dirichlet-to-Neumann maps.

A natural question is whether all matrices with the same map are related as above, via some diffeomorphism identical at the boundary. This question is generally open, but there are some positive answers. We will mention these answers in the multidimensional case, where they are obtained for real-analytic $a$. Now there are some complete answers in the two-dimensional case described in section 5.5.

Lee and Uhlmann [LeU] proved this conjecture for analytic $a_j$ in the multidimensional case under additional "convexity" assumptions.

**Theorem 5.4.2**. *Let $n \geq 3$. Let $a_1, a_2$ be two real analytic (in $\overline{\Omega}$) conductivity matrices. Let $\Omega$ be simply connected and $\partial\Omega$ analytic and strongly convex with respect to the Riemannian metrics generated by these matrices.*

*If $\Lambda_1 = \Lambda_2$, then there is a real analytic diffeomorphism $\Psi$ of $\overline{\Omega}$ onto itself such that $a_1$ and $a_2$ are related by the formula* (5.4.14).

The proof of Lee and Uhlmann relies on calculation of the full symbol of the pseudo-differential operator $\Lambda_j$ in boundary normal coordinates $\{y_1, \dots, y_n\}$, where the first $n-1$ components are (local) coordinates on $\partial\Omega$, and $y_n$ is the Riemannian distance to $\partial\Omega$. In these coordinates the Laplace-Beltrami operator $(-i\partial_n^2 + iE(x)(-i\partial_n) + Q(x, -i\partial'))$ corresponding to the metric $a^{jk}dx_j dx_k$ admits the factorization

$$(-i\partial_n + iE(x) - iA(x, -i\partial'))(-i\partial_n + iA(x, -i\partial')),$$

and the restriction of $A$ at $\partial\Omega$ and the Dirichlet-to-Neumann operator are equal up to a natural scalar factor (modulo a smoothing operator). So it is possible to express the full Taylor coefficients of $a$ at $\partial\Omega$. Now, when $a_1$ and $a_2$ produce the same $\Lambda_1$, $\Lambda_2$, they are equal near $\partial\Omega$ up to diffeomorphism, which changes normal coordinates corresponding to $a_1$ to normal coordinates corresponding to $a_2$. By using a convexity condition, it is possible to extend this diffeomorphism onto $\Omega$.

We observe that the complete asymptotic expansion of the symbol of $\Lambda$ in the anisotropic case given by Lee and Uhlmann [LeU] generalizes the results of Kohn and Vogelius [KoV1] for isotropic conductivities.

Finally we will discuss identification of a Riemannian manifold $(\Omega, g)$ with the metric $g$. This manifold posesses the Laplace-Beltrami operator $-\Delta_g$. In local coordinates the Riemannian metric $g$ is related to the elliptic (conductivity) equation via the equality $g_{jk} = (deta)^{1/(2-n)}a^{jk}$. The Laplace-Beltrami operator $-\Delta_g$ in these coordinates is defined as

$$\Delta_g u = det(g)^{-1/2} \sum \partial_j((detg)^{1/2} g_{jk}\partial_k u).$$

The (local) Dirichlet-to Neumann operator $\Lambda_\Gamma(g)$ ( at $g$) of this manifold from the part $\Gamma$ of its boundary is defined as the operator mapping the solution to the Laplace-Beltrami equation $-\Delta_g u = 0$ in $\Omega$ with the Dirichlet data $u = g_0$ on $\partial\Omega$, $g_0 = 0$ outside $\Gamma$ into the conormal derivative $\partial_{\nu(g)}u$ on $\Gamma$.

**Theorem 5.4.3**. *Let $3 \leq n$ and $\Omega$ be a real-analytic manifold with a real-analytic Riemannian metric $g$ with compact nonempty boundary.*

*Then the partial Dirichlet-to Neumann map uniquely identifies the Riemannian manifold $(\Omega, g)$ up to an isometry.*

This result belongs to Lassas and Uhlmann [LaU] (compact $\Omega$) and in more generality to Lassas, Taylor, and Uhlmann [LaTU]. As above, in their proofs they utilize boundary normal coordinates near $\Gamma$ and the analytic continuation from $\Gamma$. A new tool in [LaU], [LaTU] is an use of Green function, of its symmetry and of its singularities.

## 5.5 The plane case

In this section we assume that all coefficients are real-valued.

**Theorem 5.5.1**. *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^2$ with connected complement $\mathbb{R}^2\setminus\overline{\Omega}$. Let $a_1, a_2$ be scalar functions in $L_\infty(\Omega)$.*

*If $\Lambda_1 = \Lambda_2$, then $a_1 = a_2$ on $\Omega$.*

In this generality this result was obtained in the recent paper of Astala and Päivärinta [AsP]. For more regular conductivities $a_j$ $H_{2,p}(\Omega)$, $1 < p$ it was proven by Adrian Nachman [N3].

The both proofs use methods of the inverse scattering pioneered by Beals and Coifman [BC1], [BC2]. We give more detail in Chapter 6, and we refer for the complete proof to the papers [AsP], [N3] (see also [N1]). We will assume that $a_j = 1$ near $\partial\Omega$, which is not a restrictive assumption because as shown in Section 5.1 the conductivity coefficient can be in a stable way determined at the boundary.

The Nachman's proof contains some constructive elements and it is worth of outlining his argument. We define $c = a^{-1/2}\Delta(a^{1/2})$. Introduce the functions $\psi(x, \zeta)$ as the solutions to the Schrödinger equation

$$(-\Delta + c)\psi = 0 \text{ on } \mathbb{R}^2,\ e^{-x\cdot\zeta}\psi - 1 \in H_{1,q},\ 1/q = 1/p - 1/2,$$

where $\zeta \cdot \zeta = 0$, $\zeta \neq 0$. For the reconstruction procedure one needs only the restrictions of these functions onto $\partial\Omega$, which can be obtained by solving the following Fredholm integral equation:

$$\psi(, k) = e^{izk} - S_k(\Lambda_c - \Lambda_0)\psi(, k) \qquad \text{on } \partial\Omega$$

in the space $H_{(1/2)}(\partial\Omega)$. Here $S_k$ is the single layer potential operator

$$S_k\psi(x) = \int_{\partial\Omega} G_k(x-y)\psi(y)d\Gamma(y)$$

with respect to the Faddeev Green's function

$$G_k(x) = e^{izk}(2\pi)^{-2}\int e^{ix\cdot\xi}(|\xi|^2 + 2k(\xi_1 + i\xi_2))^{-1}d\xi.$$

Let us define

$$t(k) = \int_{\partial\Omega} e^{izk}(\Lambda_c - \Lambda_0)\psi(, k)d\Gamma.$$

Solve the differential equation

$$\partial\mu/\partial\overline{k} = (4\pi k)^{-1}e_{-k}(x)\overline{\mu}, \quad k \neq 0 \text{ where } e_k(z) = \exp(2i\Re z)$$

or equivalently, the integral equation

$$\mu(x, k) = 1 + (8\pi^2 i)^{-1}\int t(\kappa)(\kappa - k)^{-1}\kappa^{-1}e_{-x}(\kappa)\overline{\mu}(x, \kappa)dk\, d\kappa.$$

Finally, the conductivity coefficient $a$ can be found from the integral formula

$$a^{1/2}(x) = 1 + (8\pi^2 i)^{-1}\int t(k)|k|^{-2}e_{-x}(k)\overline{\mu}(x, k)dkd\overline{k}$$

or from the formula

$$a(x) = \lim_{k\to 0}\mu^2(x, k).$$

The result of Astala and Päivärinta [AsP] which completes solution of the uniqueness problem for the real-valued conductivity from the Dirichlet-to-Neumann map at the whole boundary makes use of some further ideas combining inverse scattering and theory of generalized analytic functions. By using conformal mappings we can assume that $\Omega$ is the unit disk $B_1$. To reduce regularity they use the device used by Alessandrini to study zeros of gradients of solutions of elliptic equations in the plane and outlined in section 3.3. Indeed, if $u \in H_{(1)}(\Omega)$ solves the conductivity equation $\operatorname{div}(a\nabla u) = 0$ in $\Omega$, then there is a function $v \in H_{(1)}(\Omega)$ such that $f = u + iv$ satisfies the Beltrami equation

$$\partial/(\partial\bar{z})f = \mu\overline{\partial f/\partial z} \text{ where } \mu = (1-a)/(1+a). \tag{5.5.1}$$

We extend $a$ outside $\Omega$ as 1, accordingly $\mu$ is extended as 0. In [AsP] they construct analogues of almost exponential solutions of (5.5.1)

$$f_\mu(z,\zeta) = e^{\zeta z}(1 + O(z^{-1})) \text{ as } |z| \to \infty.$$

Let

$$h_+ = 1/2(f_\mu + f_{-\mu}),\ h_- = 1/2(\bar{f}_\mu - \bar{f}_{-\mu})$$

and

$$u_1 = h_+ - ih_-,\ u_2 = i(h_+ + ih_-).$$

For $z_0$ outside $\overline{\Omega}$ we can write

$$u_1(z,\zeta) = t_{11}u_1(z_0,\zeta) + t_{12}u_2(z_0,\zeta),$$

$$u_2(z,\zeta) = t_{21}u_1(z_0,\zeta) + t_{22}u_2(z_0,\zeta).$$

A crucial step of proofs in [AsP] is to use the Beals-Coifman method to show that $\Lambda$ uniquely determines so- called transition matrix $(t_{jk})$. It can be shown that $\Lambda$ uniquely determines $f_\mu$, $f_{-\mu}$ outside $\Omega$. Now uniqueness of the transition matrix guarantees that almost exponential solutions (5.5.1) and hence the coefficient $\mu$ are uniquely determined inside $\Omega$.

Completeness of products of the two-dimensional Schrödinger equation is still unproven, and so is uniqueness of $c$ entering the equation (4.0.1) with the given Dirichlet-to-Neumann map. However, there are partial answers due to work of Isakov and Nachman [IsN] and Sun and Uhlmann [SuU1], [SuU2].

We denote by $\lambda_1$ the first eigenvalue of the Dirichlet problem for the Laplace operator in $\Omega$.

**Corollary 5.5.2**. *Let $c \in L_p(\Omega)$. Assume that $\partial\Omega \in C^2$ and*

$$c > -\lambda_1 \text{ on } \Omega. \tag{5.5.2}$$

*Then the Dirichlet-to-Neumann map $\Lambda$ for $-\Delta + c$ uniquely determines $c$.*

PROOF. Let $w$ be a solution to the Dirichlet problem

$$-\Delta w + cw = 0 \text{ in } \quad w = 1 \text{ on } \partial\Omega.$$

Because of condition (5.5.2) this solution exists, is unique, and according to Theorem 4.1 is in $H_{1,p}(\Omega)$. In addition, by embedding theorems, $w \in C(\overline{\Omega})$, and as follows from the above remarks, $w > 0$ in $\Omega$. Let us define $a = w^2$. By using embedding theorems again we find that $a \in H_{2,p}(\Omega)$ as well.

We introduce $v$ as $u/w$. Then the Schrödinger equation is transformed into the conductivity equation $\operatorname{div}(a\nabla v) = 0$ in $\Omega$. The Dirichlet-to-Neumann map for this equation is uniquely determined by the map for the original Schrödinger equation. Indeed, let $v = g$ on $\partial\Omega$. Since $w = 1$ on $\partial\Omega$, we have $u = g$ on $\partial\Omega$. In addition,

$$a\partial_\nu v = a/w\partial_\nu u - a/w^2\partial_\nu wu = \partial_\nu u - \partial_\nu wu \text{ on } \partial\Omega.$$

The equation for $w$ coincides with the equation for $u$, so the Dirichlet-to-Neumann map for the Schrödinger equation uniquely determines $\partial_\nu w$. In sum, we conclude that $a\partial_\nu v$ is uniquely determined, and we are given the Dirichlet-to-Neumann map for the conductivity equation. Then $a$ is uniquely determined by Theorem 5.5.1, and so is $w = a^{1/2}$. Then $c = \Delta w/w$.

The proof is complete. □

We believe that there is global uniqueness for $c$ without the assumption (5.5.2). At present there are several ideas how to prove it, but no proof is available. We give only the following partial result of Sun and Uhlmann [SuU1] for real-valued $c$.

**Theorem 5.5.3** (Generic Uniqueness). *There exists an open and dense set $\mathcal{C}$ in $H_{1,\infty}(\Omega) \times H_{1,\infty}(\Omega)$ such that if $(c_1, c_2) \in \mathcal{C}$ and $\Lambda_1 = \Lambda_2$, then $c_1 = c_2$.*

An idea of a proof (for equation (5.2.1) is again to use almost exponential solutions (5.3.3) with the choice $\zeta(1) = 1/2(\xi + i\eta)$, $\zeta(2) = 1/2(\xi - i\eta)$, where $\eta \in \mathbb{R}^2$ is orthogonal to $\xi$ and has the same length as $\xi$. In the plane case these solutions are well-defined only when $|\xi| > C(\|c_j\|_\infty(\Omega))$. Observe that Sylvester and Uhlmann [SyU1] proved that

$$w(x;\zeta(j)) = a(x;j)/(\eta_2 + i\eta_1) + b(x;\eta, j), \tag{5.5.3}$$

where $|b| \leq C|\eta|^{-2}$. Let us define the operator

$$A(c_1, c_2)c(x) = F\int_\Omega c(x)e^{ix\cdot\xi}(1 + w(x;\xi))dx, \tag{5.5.4}$$

where $F$ is the inverse Fourier transformation and $w = w_1 + w_2 + w_1w_2$ when $|\zeta| > C$ and $w = -1$ otherwise. Here $w_j$ is the solution to equation (5.3.11). Observe that it depends on $c_j$ analytically. By using the representation (5.5.3) it is not difficult to show that

$$A(c_1, c_2)c = c + Kc, \tag{5.5.5}$$

where $K$ is a compact (smoothing) operator in $H_{1,\infty}(\Omega)$.

Letting $c = c_2 - c_1$ and using the orthogonality relations (5.2.2) with $u_j = \exp(ix \cdot \zeta(j))(1 + w_j)$ as well as the definition (5.5.4) of the operator $A$, we obtain that $c + Kc = 0$. Now one can employ the analytic Fredholm theory to conclude that $I + K$ is invertible for "almost all" $c_1, c_2$ in the sense of Theorem 5.5.3.

Another global result of Sun and Uhlmann is about uniqueness of recovery of discontinuities [SuU2].

**Theorem 5.5.4**. *Let $c_j \in L_\infty(\Omega)$. If $\Lambda_1 = \Lambda_2$, then $c_2 - c_1 \in C^\alpha(\overline{\Omega})$ for any $\alpha \in [0, 1)$.*

We will explain the basic idea of the proof. Arguing as above, we obtain $c = Kc$, where $c = c_2 - c_1$. By (5.5.3), $K$ is a smoothing operator. More careful study shows that it is smoothing in Hölder spaces, and this completes the proof. We observe that we only tried to explain an idea of the proof in [SuU2] that is quite ingenious and that uses in fact a different approach based on a special operator from scattering theory.

Now we turn to identification of several coefficients. We are considering the general elliptic equation (5.4.1).

**Theorem 5.5.5**. *Let two equations (5.4.1) with the coefficients $a_j = 1, b_j, c_j, j = 1, 2$ posess the same Dirichlet-to Neumann maps.*

*a) Let $b_j \in H_{3,\infty}(\Omega)$, $\Re b_j = 0$, $\Im c_j = 0$, and $c_j \in H_{1,\infty}(\Omega)$.*

*There exists an open and dense set $\mathcal{C}$ in $H_{1,\infty}(\Omega)$ such that for any $c \in \mathcal{C}$ one has a neighborhood $V$ and a positive number $\varepsilon(c, \Omega)$ such that when $\| curl\, b_j \|_{2,\infty}(\Omega) < \varepsilon$ and $c_j \in V$, then*

$$curl\, b_1 = curl\, b_2 \text{ and } 4c_1 + b_1 \cdot b_1 - 2\,\mathrm{div}\, b_1 = 4c_2 + b_2 \cdot b_2 - 2\,\mathrm{div}\, b_2 \text{ on } \Omega. \tag{5.5.6}$$

*b) Let $b_j \in L_p(\Omega)$ for some $p > 2$ and $c_j = 0$.*

*Then $b_1 = b_2$ in $\Omega$.*

Part a) of this theorem is due to Sun and the global uniqueness of part b) is proved by Cheng and Yamamoto [CheY2]. Cheng and Yamamoto also use theory of Beals and Coifman [BC1] with additions of Sung [Sun].

Now we discuss anisotropic equations $div(a\nabla u) = 0$.

**Theorem 5.5.6**. *Let $\Omega$ be a simply connected domain in $\mathbb{R}^2$, $\partial\Omega \in C^3$. Let $a_1, a_2$ be two (symmetric) $L_\infty(\Omega)$ conductivity matrices.*

*If $\Lambda_1 = \Lambda_2$, then there is a $H_{1,2}(\Omega)$-diffeomorphism $\Psi$ identical on $\partial\Omega$ such that $a_2$ is related to $a_1$ via* (5.4.14).

The proof of Sylvester [Sy1] (given under the assumption that $|\ln(\det a_j)|_3(\Omega)$ is small) is based on introducing isothermal coordinates by mapping $\Omega$ onto the unit disk so that the conductivity matrix $a_j$ is proportional to the identity (isotropic case). The next and crucial step of the proof is to show that the first mapping composed with the inverse to the second one is conformal on the unit disk. The application of the uniqueness theorem 5.5.1 makes Sylvester's argument a global one. In this most general and natural form of Theorem 5.5.5 was proven by Astala, Lassas, and Päivärinta [AsLP]. We cannot go into further detail, referring rather to [AsLP], [N3] and [Sy1]. We emphasize only that this proof is "two-dimensional."

Now we turn to identification of two-dimensional Riemannian manifolds $(\Omega, g)$ with boundaries from their Dirichlet-to Neumann map $\Lambda$

**Theorem 5.5.7**. *A two-dimensional $C^\infty$-smooth Riemannian compact manifold is uniquely identified up to an isometry by its local Dirichlet-to Neumann map.*

The first proof of this result belongs to Lassas and Uhlmann [LaU] who based it on use of special conformal coordinates and on uniqueness of the continuation for harmonic functions. The more recent proof of Belishev [Be4] uses completely different ideas from the Gelfand theory of Banach Algebras of complex analytic functions and ideals of these algebras.

## 5.6 Nonlinear equations

The results of Sylvester and Uhlmann on the Schrödinger equation have been extended onto the semilinear elliptic equations

$$-\Delta u + c(x, u) = 0 \quad \text{in } \Omega. \tag{5.6.1}$$

Let us introduce the following Caratheodory type conditions:

$$|c(, u)| + |\partial_u c(, u)| \leq \phi \text{ when } |u| < U, \text{ where } \phi \in L_p(\Omega);$$
$$p = +\infty \text{ when } n \geq 3,\ p > 1 \text{ when } n = 2;$$
$$\partial_u c(x, u) \text{ is continuous with respect to } u \text{ for any } x \in \Omega; \tag{5.6.2}$$

and the semipositivity condition

$$\partial_u c > -\lambda_1 + \varepsilon \tag{5.6.3}$$

for some positive $\varepsilon$, where $\lambda_1$ is the first eigenvalue to the Dirichlet problem for the Laplace operator in $\Omega$.

First, we will discuss solvability of the Dirichlet problem for the semilinear equation (5.6.1). This is easier to do under the condition $\partial_u c \geq 0$, but we think that condition (5.6.3) is much more useful in applications, because it allows solutions that can be small near $\partial\Omega$ and large inside $\Omega$. Exactly in this situation it is very interesting to use boundary measurements, because the interior is not accessible for measurements.

Let us consider the linear Dirichlet problem $-\Delta w + (-\lambda_1 + \varepsilon/2)w = 0$ in a slightly larger $C^\infty$-domain $\Omega_\bullet$, $w = 1$ on $\partial\Omega_\bullet$. We choose $\Omega_\bullet$ so that it contains $\overline{\Omega}$ and that the first eigenvalue for $-\Delta$ in $\Omega_\bullet$ is greater than $\lambda_1 - \varepsilon/2$. There is a unique solution $w \in C^\infty(\overline{\Omega_\bullet})$ to this problem, and this solution is positive on $\Omega_\bullet$. To prove positivity one can consider the family of the same Dirichlet problems with $-\lambda_1$ replaced by $-\tau\lambda_1$, $\tau \in [0, 1]$. By the definition of the first eigenvalue, all these problems have unique solutions. By the maximum principle a solution is positive on $\Omega_\bullet$ when $\tau = 0$. If it is not positive for $\tau = 1$, then by continuity reasons there is $\tau_0$ such that the corresponding solution is nonnegative on $\Omega_\bullet$ and is zero at some point of $\Omega_\bullet$. Since the boundary data are 1, this point is in $\Omega_\bullet$, and we have a contradiction because a nonnegative solution to an elliptic equation that is zero at a point of a domain must be zero by the Harnack inequality. To solve the semilinear problem we will make use of the substitution $u = wv$. Then, for the new function $v$ to be found, we obtain the differential equation

$$Lv + c^+(x, v) = 0 \text{ in } \Omega, \tag{5.6.1$^+$}$$

where $L = -w\Delta - 2\nabla w \cdot \nabla$ and $c^+(x, v) = (\lambda_1 - \varepsilon/2)wv + c(x, wv)$. From condition (5.6.3) it follows that $\partial_v c^+ \geq 0$.

Now we will prove that for any Dirichlet boundary data $g_0 \in C^2(\partial\Omega)$ there is a unique solution $u(; g_0) \in C^1(\overline{\Omega})$ to the Dirichlet problem (5.6.1), (4.0.2), so again we have the well-defined Dirichlet-to-Neumann map. It suffices to prove this for equation (5.6.1$^+$) instead of the original equation.

Since $g_0$ is continuous on $\partial\Omega$, it is bounded ($|g_0| \leq C$) there. We change $c^+(x, v)$ for $|v| > 2C$, preserving all its properties but so that in addition $\|c^+(, v)\|_p(\Omega)$ is bounded uniformly with respect to $v$. The linear Dirichlet problem $Lv = c$ in $\Omega$, $u = g$ on $\partial\Omega$ has a unique solution $L^{-1}c$ for any $c \in L_p(\Omega)$. Moreover, $L^{-1}$ is a compact operator from $L_p(\Omega)$ into $L_\infty(\Omega)$. To deduce the solvability (of the linear Dirichlet problem) from the standard elliptic theory, we fix a ball $B$ containing $\overline{\Omega}$, extend $c^+$ as zero outside $\Omega$, and let $v_0$ be the solution be the solution to the Dirichlet problem $Lv_0 = c$ in $B$, $v_0 = 0$ on $\partial B$. By the Schauder-type estimates of Theorem 4.1, we have $\|v_0\|_{2,p}(B) \leq C\|c\|_p(\Omega)$. The Sobolev embedding theorems and conditions (5.6.2) guarantee that the operator $c \to v_0$ is compact from $L_p(\Omega)$ into $L_\infty(\Omega)$. Let $v_1$ be a solution to the new Dirichlet problem $Lv_1 = 0$ in $\Omega$, $v_1 = g - v_0$ on $\partial\Omega$. By maximum principles the operator $v_0 \to v_1$ is continuous from $L_\infty(\Omega)$ into the same space, so the operator $c \to v_1$ from $L_p(\Omega)$ into $L_\infty(\Omega)$ is compact as the composition of compact and continuous operators. Since $v = v_0 + v_1$, the resulting operator is compact as well. Let us consider the equation $v = -L^{-1}c^+(, v)$ in the ball $\|v\|_\infty(\Omega) \leq C_1$. When $C_1$ is large enough, the operator from the left side of the equation maps this ball into itself. Moreover, this mapping is compact. By the Schauder-Tikhonov fixed point theorem there is a solution $v$ to this equation. By the construction, $v$ solves equation (5.6.1$^+$) with changed $c^+$. By the comparison principle, which is valid due to the condition $\partial_v c^+ \geq 0$ for the changed equation, we have $|v| \leq C$, and for such $v$ we have the old equation. Our solution $v$ will be in $H_{(1)}(\Omega)$, as follows from the linear theory when we consider $c^+$ as the right side of the elliptic differential equation.

There are several other ways to prove solvability of the Dirichlet problem for the semilinear equation (5.6.1), for example the continuity method.

When one has two solutions $v_1, v_2$ to equation (5.6.1$^+$) with the Dirichlet data $g^+_{0,1}$, $g^+_{0,2}$, then subtracting the equation for $v_1$ from the equation for $v_2$ and using that

$$c^+(x, v_2) - c^+(x, v_1) = c_*(x)(v_2 - v_1),$$
$$c_*(x) = \int_0^1 \partial_v c^+(x, (v_1 + s(v_2 - v_2))(x))ds,$$

one obtains for the difference a linear elliptic equation with maximum principle because $\partial_v c \geq 0$, and therefore $c_* \geq 0$. Hence, we have uniqueness of a solution of the Dirichlet problem.

Our substitution and the maximum principle for $v$ imply that

$$C \min g_0 \leq u \leq C \max g_0 \tag{5.6.4}$$

for a solution to the Dirichlet problem for the original equation (5.6.1) with the Dirichlet data $g_0$ on $\partial\Omega$, provided that $\min g_0 \leq 0 \leq \max g_0$.

We will need the following notation $E = \{(x, u) : u_*(x) < u < u^*(x)\}$, where $u_*(x)$ is the infimum and $u^*(x)$ is the supremum of $u(x; \theta)$ over $\theta \in \mathbb{R}$. The following result was obtained by Isakov and Sylvester [IsSy] when $n \geq 3$ and by Isakov and Nachman [IsN] when $n = 2$.

**Theorem 5.6.1**. *Let $c$ satisfy the assumptions* (5.6.2), (5.6.3) *and moreover,*

$$c(x, 0) = 0. \tag{5.6.5}$$

*Then $c$ on $E$ is uniquely identified by the Dirichlet-to-Neumann map of equation* (5.6.1).

PROOF. Let $c^*(x; \theta) = \partial_u c(x; u(x; \theta))$. Let us consider the linear differential equation

$$-\Delta v + c^* v = 0 \text{ in } \Omega \tag{5.6.6}$$

and its Dirichlet-to-Neumann map $\Lambda^*$. In fact, this equation is the linearization of equation (5.6.1) around the solution $u(; \theta)$. □

**Lemma 5.6.2**. *$\Lambda$ uniquely determines $\Lambda^*$ for any $\theta$. Moreover, $v = \partial_\theta u(; \theta)$ exists and it satisfies equation* (5.6.6) *and has the Dirichlet data*

$$\partial_\theta u(; \theta) = 1 \textit{ on } \partial\Omega. \tag{5.6.7}$$

PROOF. The regularity assumptions (5.6.2) and Taylor's formula in the integral form yield

$$c(, \theta_1 + \tau_1) - c(, \theta_1) - \partial_u c(, \theta_1)\tau_1 = r(; \theta_1, \tau_1)\tau_1,$$

with

$$r(x; \theta_1, \tau_1) = \int_0^1 (\partial_u c(x, \theta_1 + s\tau_1) - \partial_u c(x, \theta_1))ds.$$

Let us consider any Dirichlet data $g_0^* \in H_{(1/2)}(\partial\Omega) \cap C(\partial\Omega)$. By subtracting two equations (5.6.1) for $u(; \theta + \tau g_0^*)$ and for $u(; \theta)$, dividing the result by $\tau$, and denoting by $v(; \tau)$ the finite difference $(u; \theta + \tau g_0^*) - u(; \theta))/\tau$, we obtain the linear differential equation

$$-\Delta v(; \tau) + (c^*(; \theta) + r^*(; \theta, \tau))v(; \tau) = 0 \text{ on } \Omega \tag{5.6.8}$$

with $r^*(x;\theta,\tau) = r(x;u(x;\theta), u(x;\theta+\tau g_0^*) - u(x;\theta))$ and the Dirichlet boundary condition

$$v(;\tau) = g_0^* \text{ on } \partial\Omega. \tag{5.6.9}$$

As observed above, this Dirichlet problem is uniquely solvable, and moreover, the known energy estimates and maximum principles (Theorems 4.1 and 4.2) combined with (5.6.4) give

$$\|v(;\tau)\|_{(1)}(\Omega) + \|v(;\tau)\|_\infty(\Omega) \leq C.$$

Let $v(;0)$ be a solution to the Dirichlet problem (5.6.8), (5.6.9) with $r^* = 0$. Subtracting the equations for $v(;\tau)$ and for $v(;0)$ and referring $r^*v$ into the right side, we obtain the Dirichlet problem for $v(;\tau) - v(;0)$,

$$-\Delta(v(;\tau) - v(;0)) + c^*(;\theta)(v(;\tau) - v(;0)) = -r^*(;\theta,\tau)v(;\tau) \text{ in } \Omega,$$
$$v(;\tau) - v(;0) = 0 \quad \text{on } \partial\Omega.$$

From the above estimate of $v(;\tau)$ and the definition of $r^*$, it follows by Theorem 4.1 that the norm $\| \ \|_p(\Omega)$ of the right side goes to zero. Indeed, $\|r^*\|_p(\Omega) \to 0$ by the formulae for $r, r^*$, conditions (5.6.2), the uniform bound on $v(;\tau)$ and the Lebesgue dominated convergence theorem. Again from Theorem 4.1 we conclude that $v(;\tau)$ converges to $v(;0)$ in $H_{(1)}(\Omega)$ as $\tau \to 0$, so $\partial_\nu v(;\tau)$ goes to $\partial_\nu v(;\tau)$ in $H_{(-1/2)}(\partial\Omega)$. Since $\Lambda$ is given, $\partial_\nu v(;\tau)$ on $\partial\Omega$ is given as well, and so is their limit. We can pass to the limit as well in equation (5.6.8), obtaining equation (5.6.6) for $v(;0)$.

Summing up, for any admissible Dirichlet data $g^*$ we are given the Neumann data for the solution $b$ to the corresponding Dirichlet problem for (5.6.6), so we are given the Dirichlet-to-Neumann map for this equation.

Letting in particular $g^* = 1$, we obtain the remaining conclusion of this lemma. □

END OF THE PROOF OF THEOREM 5.6.1. From Theorems 5.2.1 and 5.5.2 and Lemma 5.6.2 it follows that the coefficient $c^*$ of the differential equation (5.6.6) with the given Dirichlet-to-Neumann map is uniquely determined. Since the linear Dirichlet problem (5.6.6), (5.6.7) is uniquely solvable, we can find the function $\partial_\theta u(;\theta)$. Using that $u(;0) = 0$ due to condition (5.6.5), we obtain

$$u(x;\theta) = \int_0^\theta \partial_\tau u(x;\tau)d\tau \tag{5.6.10}$$

As observed in the discussion of the Dirichlet problem, under condition (5.6.3) the solution $\partial_\theta u(;\theta)$ to the Dirichlet problem (5.6.6), (5.6.7) satisfies the inequality $\partial_\theta u > \varepsilon > 0$ with $\varepsilon$ depending only on $\Theta$, provided that $|\theta| \leq \Theta$. So for any $x \in \Omega$ and any $u \in (u_*(x), u^*(x))$ we can find unique $\theta$ such that $u = u(x;\theta)$. Then $\partial_u c(x,u) = \partial_u c(x, u(x,\theta))$ is known, and so is $c(x,u)$ due to condition (5.6.5).

The proof is complete. □

**Corollary 5.6.3**. *Assume that in addition to the conditions of Theorem* 5.6.1 *we have* $\partial_u c \leq C$.

*Then* $c$ *on* $\Omega \times \mathbb{R}$ *is uniquely determined by* $\Lambda$.

PROOF. It suffices to observe that under the conditions of Corollary 5.6.3 we have $E = \Omega \times \mathbb{R}$. The function $v = \partial_\theta u(;\theta)$ solves the Dirichlet problem (5.6.6), (5.6.7) with the coefficient $c^*$ bounded from below by $-\lambda_1 + \varepsilon$ and from above by $C$. By using the remarks on elliptic equations under condition (5.6.3) and comparison theorems for solutions of the Dirichlet problem (Theorem 4.2), we conclude that $u_\bullet \leq \partial_\theta u(x;\theta)$, where $u_\bullet$ is a solution to the Dirichlet problem

$$-\Delta u_\bullet + C u_\bullet = 0 \text{ in } \Omega, \quad u_\bullet = 1 \text{ on } \partial\Omega.$$

By maximum principles $\varepsilon < u_\bullet$, so $\partial_\theta u > \varepsilon$ (which does not depend on $\theta$) as well. This implies that $u_*(x) = -\infty$ and $u^*(x) = +\infty$. The proof is complete. □

We observe that for the equation $-\Delta u + u^{2k+1} = 0$ the range of a solution at an interior point of $\Omega$ is bounded from above and below uniformly with respect to Dirichlet boundary data, so one cannot expect recovery of $a(x,u)$ for $x \in \Omega$, $u \in \mathbb{R}$ without the second condition $\partial_u a \leq C$. Indeed, consider the Dirichlet problem

$$(5.6.11) \quad -u'' + u^{2k+1} = 0 \qquad \text{in } \Omega = (-1,1), \quad u(-1) = u(1) = \theta > 0.$$

As shown above, a solution $u(;\theta)$ to this problem exists and is unique. Since the problem is invariant with respect to the substitution $x \to -x$, so is its solution. In other words, $u(x) = u(-x)$. In particular, $u'(0) = 0$. Multiplying the differential equation (5.6.11) by $u'$ and integrating over the interval $(0,x)$, we find that $(k+1)u'^2(x) = u^{2k+2}(x) - u^{2k+2}(0)$, so $u(x) > u(0)$ when $x > 0$. Solving for $u'$ and integrating this ODE of first order by separating variables, we find

$$x = (k+1)^{1/2}/u(0) \int_1^{u(x)/u(0)} \left(s^{2k+2} - 1\right)^{-1/2} ds.$$

From this relation we conclude that $u(x)$ is a monotone function of $x$ on $(0,1)$, so $u(0) \leq u(x)$. Letting $x = 1$, we obtain

$$u(0) = (k+1)^{1/2} \int_1^{\theta/u(0)} (s^{2k+2} - 1)^{-1/2} ds$$
$$< (k+1)^{1/2} \int_1^{+\infty} (s^{2k+2} - 1)^{-1/2} ds < +\infty.$$

So for this differential equation $u^*(0) < +\infty$, and $E$ is not $\Omega \times \mathbb{R}$.

That global uniqueness holds for the nonlinear conductivity equations

$$(5.6.12) \qquad \operatorname{div}(a(,u)\nabla u) = 0 \text{ in } \Omega$$

was shown by Sun [Su2].

**Theorem 5.6.4**. *Assume that a nonlinear conductivity coefficient* $a(x,u) \in C^2(\overline{\Omega} \times (-U,U))$ *for any number* $U$.

*Then the Dirichlet-to-Neumann map for the nonlinear elliptic equation* (5.6.12) *uniquely determines a on* $\Omega \times \mathbb{R}$.

The proof in [Su2] uses the linearization scheme described above. Observe that equation (5.6.12) has constant solutions $u = \theta$, so one can linearize at these solutions. As suggested by (formal) differentiation of equation (5.6.12) with respect to the parameter $\theta$ (we have $\nabla\theta = 0$), the linearized equations will be the linear conductivity equations with the coefficients $a(x, \theta)$. As in the proof of Theorem 5.6.1, one can show that the initial Dirichlet-to-Neumann map uniquely determines the Dirichlet-to-Neumann map for the linearized equation. So Theorems 5.2.1 and 5.4.1 guarantee that $a(x, \theta)$ is uniquely determined.

Some partial results are available for anisotropic case, i.e. when $a = a(x, u)$ in the equation (5.6.12) is a symmetric positive matrix.

**Theorem 5.6.5**. *Let* $\Omega$ *be a bounded domain in* $\mathbb{R}^n$ *of class* $C^{3+\lambda}$. *Let* $a_1, a_2$ *be quasilinear coefficient matrices in* $C^{2+\lambda}(\overline{\Omega})$ *with* $0 < \lambda < 1$ *such that the quasilinear equations (5.6.12) with these matrix coefficients have the same Dirichlet-to Neumann maps.*

*a) If* $n = 2$, *then there is a* $C^{3+\lambda}$*-diffeomorphism* $\Psi$ *of* $\overline{\Omega}$ *onto itself which is the identity on* $\partial\Omega$ *such that two quasilinear matrix coefficients are related via (5.4.14).*

*b) If* $3 \leq n$, $\partial\Omega$ *and* $a_j(x, u)$ *are (real-) analytic, then there is an (real-) analytic diffeomorphism* $\Psi$ *of* $\overline{\Omega}$ *onto itself which is the identity on* $\partial\Omega$ *such that two quasilinear matrix coefficients are related via (5.4.14).*

This result is due to Sun and Uhlmann [SuU3]. It is obtained by combining available results from theory of linear equations with the linearization method. Linearizing at constant solutions $u = \theta$ and considering the Dirichlet-to Neumann map for the linearized equations

$$div(a_j(x, \theta)\nabla v) = 0 \text{ in } \Omega$$

we obtain then that two coefficients can be obtained from each other by the formula (5.4.14) with a diffeomorphism $\Psi(x, u)$ and the basic step of the proof is to show that $\Psi$ does not depend on $u$. For this goal they used in [SuU3] the second linearization.

Finally we report some recent result on identification of the term $c(u, p)$ of the quasilinear equation

$$-\Delta u + c(u, \nabla u) = 0 \text{ in } \Omega$$

from its local Dirichlet-to-Neumann map $\Lambda_\Gamma$ from a part $\Gamma$ of $\partial\Omega \in C^3$. It was mentioned in section 5.4 that the term $c(x, u, \nabla u)$ can not be uniquely identified even for linear equations. In our case there is a complete uniqueness result [Is15].

**Theorem 5.6.6**. *Let* $n = 3$. *Let* $c \in C^3(\mathbf{R}^4)$ *satisfy the conditions*

$$0 \leq \partial_u c(u, p),$$

*and*

$$|c(u, p)| + |\partial_u c(u, p)| \leq C(U)(1 + |p|)^{2-\delta}, \ |\nabla_p c(u, p)| \leq C(U)(1 + |p|)$$

*for some* $\delta > 0$.

*Then the local Dirichlet-to-Neumann map* $\Lambda_\Gamma$ *uniquely determines* $c$ *on* $\mathbb{R}^4$.

A proof in [Is15] is based on boundary reconstruction which is stable as observed in section 5.1 and on the linearization method. So one can expect a very strong (conditional Hölder or Lipschitz stability) in this problem and hence very efficient reconstruction algorithms.

## 5.7 Discontinuous conductivities

In this section we will use the results of measurements implemented on a open nonempty part $\Gamma$ of the boundary of $\Omega$. We introduce the local Dirichlet-to-Neumann map $\Lambda_\Gamma$, which is defined on Dirichlet data $g \in H_{(1/2)}(\partial\Omega)$ that are 0 on $\partial\Omega\backslash\Gamma$ and maps them into the Neumann data only on $\Gamma$. In other words, we initiate our solution from $\Gamma$ and measure the response only on $\Gamma$ as well. In this section we consider equation (4.0.1) with $b = 0, c = 0, f = 0$ in $\mathbb{R}^n$, $n \geq 2$.

The piecewise smooth conductivity $a$ is quite interesting from both theoretical and practical points of view. We call $a$ piecewise $C^2$-smooth (analytic) when there is a subdivision of $\Omega$ into a finite number of its mutually disjoint subdomains $\Omega(k; a)$, $k = 1, \ldots, K(a)$ with piecewise $C^1$-smooth boundaries such that $a$ has a $C^2$-(analytic) continuation onto a neighborhood of any $\overline{\Omega}(k; a)$.

**Theorem 5.7.1**. *Let* $a_1, a_2$ *satisfy one of the following three conditions* (1) *They are piecewise analytic in* $\Omega$ *and their analyticity domains* $\Omega(k; a_j)$ *have piecewise analytic boundaries.* (2) *They are constant on* $\Omega(k; a_j)$. (3) $a_j = a_0 + \chi(D_j)a_j^\bullet$, *where* $a_0$ *is a given and the* $a_j^\bullet$ *are unknown* $C^2(\overline{\Omega})$*-functions,* $a_j^\bullet \neq 0$ *on* $\partial D_j$ *and* $D_j$ *is an unknown open subset of* $\Omega$ *with Lipschitz boundary and with connected* $\Omega\backslash\overline{D}_j$. *Let* $\Lambda_{j,\Gamma)}$ *be the local Dirichlet-to Neumann map for the equation* $div(a_j \nabla u) = 0$ *in* $\Omega$.

*Let* $\Lambda_{1,\Gamma} = \Lambda_{2,\Gamma}$. *Then in the all three cases we have* $a_1 = a_2$.

Case (1) has been considered by Kohn and Vogelius in the paper [KoV2], where the basic idea is to use Theorem 5.1.1 or Exercise 5.1.3 and analyticity of $a_j$ in subdomains to determine coefficients in subdomains adjacent to $\partial\Omega$, then to exploit the Runge property to conclude that the Dirichlet-to-Neumann maps coincide for smaller subdomains, and then to repeat the first step.

Case (2) follows from the results of the paper of Druskin [Dr1]. The basic tool there is the relation

$$a^{-1}(x) = 4\pi \lim_{y \to x} |x - y| G(x, y), \ n = 3 \tag{5.7.1}$$

where $a(x)$ is $(a(x;k)+a(x;m))/2$ when $x$ is on a smooth part of $\partial\Omega(k)\cap\partial\Omega(m)(a(x;k)$ is the limit of $a(y)$ when $y$ goes to $x$ from inside of $\Omega(k))$ and is just $a(x)$ otherwise. Here $G(x, y)$ is a fundamental solution to equation (4.0.1) with the pole at $y$.

Case (3) is considered in the paper [Is3]. The idea there is to use singular solutions to the differential equations (4.0.1) in the orthogonality relations (5.0.4) to obtain a contradiction when one assumes that a point $0 \in D_2\backslash\overline{D_1}$. The implementation is similar to the outline of the proof of Theorem 5.1.1. However, it is more complicated because one has to use the Runge property and the topological assumptions on $D_j$. The result is unknown without these assumptions (say, for annular $D_j$).

We describe the basic steps of this proof.

Generally, it is possible that the set $\Omega\backslash(\overline{D_1}\cup\overline{D_2})$ is not connected, so there is more than one connected component of it. Let $\Omega_0$ be those connected components whose boundary contains $\Gamma$. Let $D_0$ be $\Omega\backslash\overline{\Omega_0}$.

**Lemma 5.7.2**. *Under the conditions of Theorem* 5.7.1 *we have*

$$\int_{D_1} a_1^\bullet \nabla v_1\cdot\nabla v_2 = \int_{D_2} a_2^\bullet \nabla v_1\cdot\nabla v_2 \tag{5.7.2}$$

*for all solutions $v_1$, $v_2$ to equations* (4.0.1) *near* $\overline{D}_0$.

PROOF. Since the solutions $u_1, u_2$ to equations (4.0.1) satisfy the same elliptic differential equation on $\Omega_0$ and they have the same Cauchy data on $\Gamma$, by the uniqueness in the Cauchy problem (Theorem 3.3.1) we have $u_1 = u_2$ on $\Omega_0$. Let $V\subset\Omega$ be any neighborhood of $\overline{D}_0$ and $v_2$ a solution to equation (4.0.1) in this neighborhood. By shrinking $V$ when necessary, we can assume that $v_2\in H_{(1)}(V)$. Subtracting the equations for $u_2$ and $u_1$ and letting $u = u_2 - u_1$, we obtain

$$\operatorname{div}(a_2\nabla u) = \operatorname{div}((\chi(D_1)a_1^\bullet - \chi(D_2)a_2^\bullet)\nabla u_1) \quad \text{in } V,$$

so the definition (4.0.3) of a (weak) solution to this equation with the test function $v_2$ gives

$$\int_\Omega a_2\nabla u\cdot\nabla v_2 = \int_\Omega(\chi(D_1)a_1^\bullet - \chi(D_2)a_2^\bullet)\nabla u_1\cdot\nabla v_2$$

because $u = 0$ near $\partial V$. On the other hand, from the same definition of a solution $v_2$ to the second equation with the test function $u$, we conclude that the left integral is zero, so we have the relation (5.7.2) with $v_1 = u_1$.

To complete the proof it suffices to extend the equality (5.7.2) onto solutions $v_1$ near $\overline{D}_0$. We denote the space of such solutions by $H$, letting $H_\Gamma$ be the space of solution to the Dirichlet problem (4.0.1), (4.0.2) with $g = 0$ on $\partial\Omega\backslash\Gamma$. If $H_\Gamma$ is not dense in $H$ with respect to the norm in $L_2(D_0)$, then by the Hahn-Banach theorem, there is $f\in L_2(D_0)$ that is not zero such that $(f, u_1)_2(D_0) = 0$ for any $u_1\in H_\Gamma$, but $(f, v_1)_2(D_0)\neq 0$ for some $v_1\in H$. We can assume that $V$ has a smooth boundary and $v_1$ is a solution in a neighborhood of $V$. Let $\Omega_\bullet$ be a Lipschitz bounded domain containing $\Omega$ that is not equal to $\Omega$ but such that $\partial\Omega\backslash\Gamma\subset\partial\Omega_\bullet$. Let $G(x, y)$ be

Green's function of the Dirichlet problem for the operator $\operatorname{div}(a_1\nabla)$ in $\Omega_\bullet$. Then by the single layer potential representation we have

$$v_1(y) = \int_{\partial V} G(x, y)\sigma(x)d\Gamma(x), x \in V$$

for some $\sigma \in L_1(\partial V)$. Since $u_1 = G(x, ) \in H_\Gamma$ when $x \in \Omega_\bullet \backslash \overline{\Omega}$, we have $(f, G(x, ))_2(D_0) = 0$. The last function $U_1(x)$ is the volume potential with density $f$ supported in $\overline{D}_0$, so it is a solution to the elliptic equation $\operatorname{div}(a_1\nabla U_1) = 0$ in $\Omega_\bullet \backslash \overline{D}_0$. Since it is zero on $\Omega_\bullet \backslash \overline{\Omega}$ and $a_1 \in C^1$ outside $D_0$, we can use uniqueness of the continuation and conclude that $U_1 = 0$ in particular on $\partial V$. By using the above representation and the Fubini theorem, we obtain

$$\begin{aligned}(f, v_1)_2(D_0) &= \int_{\partial V} \sigma(x)\left(\int_{D_0} f(y)G(x, y)dy\right) d\Gamma(x)\\ &= \int_{\partial V} \sigma(x)U_1(x)d\Gamma(x)\\ &= 0,\end{aligned}$$

which contradicts the choice of $v_1$. So we can claim that any solution near $\overline{D}_0$ can be approximated by solutions $u_1$ in $L_2(D_0)$. Applying this result to a domain $D_{\bullet_0}$ containing the closure of $D_0$ and using interior Schauder-type estimates for elliptic equations in the divergent form ($L_2$-convergence of solutions in $D_{\bullet_0}$ implies $H_{(1)}$-convergence in $D_0$, by Theorem 4.1), we complete the proof. □

We return to the proof of Theorem 5.7.1.

First we will show that $D_1 = D_2$. Let us assume the opposite: $D_2$ is not contained in $D_1$. By using that the $\Omega \backslash \overline{D}_j$ are connected we can find a point of $D_2 \backslash \overline{D}_1$ that is in $\partial D_0$. We may assume that this point is the origin. We choose the ball $B$ so that its closure is contained in $\Omega$ and does not intersect $\overline{D}_1$. By the extension theorem there is a function $a_4 \in C^1(B \cup D_2)$ that agrees with $a_2$ on $D_2$. Let us extend $a_4$ on the rest of $\Omega$ as $a_0$. We claim that

$$\int_{D_1} a_1^\bullet \nabla v_1 \cdot \nabla v_4 = \int_{D_2} a_2^\bullet \nabla v_1 \cdot \nabla v_4 \tag{5.7.3}$$

for any solution $v_4$ to the equation $\operatorname{div}(a_4\nabla v_4) = 0$ near $\overline{D}_0$. To prove this we approximate $v_4$ in $H_{(1)}(D_0)$ by functions $v_{4k}$ solving the equation $\operatorname{div}(a_{4k}\nabla v_{4k}) = 0$ near $\overline{D}_0$. The volumes of the sets $B_k = \{y \in B\backslash D_0 : \operatorname{dist}(y, D_0) < 1/k\}$ go to zero when $k \to \infty$ because $\partial D_0 \cap B$ is Lipschitz. We define $a_{4k}$ to be $a_4$ on $\Omega \backslash B_k$ and $a_2$ on $B_k$. As above, we can assume that $v_4$ solves the differential equation near $\overline{V}$ (a neighborhood of $\overline{D}_0$) with the smooth boundary. We define $v_{4k}$ as a solution to the Dirichlet problem $\operatorname{div}(a_{4k}\nabla v_{4k}) = 0$ in $V$, $v_{4k} = v_4$ on $\partial V$. By subtracting the differential equations for $v_{4k}$ and $v_4$ we obtain $\operatorname{div}(a_{4k}\nabla(v_{4k} - v_4)) = -\operatorname{div} f_k$, where $f_k$ is the vector field $(a_{4k} - a_4)\nabla v_4$. In addition, $v_{4k} - v_4 \in \mathring{H}_{(1)}(V)$. Since $\nabla v_4 \in L_2(V)$, we have $f_k \to 0$ in $L_2(V)$, so the well-known estimates for the elliptic equations in the divergent form of Theorem 4.1 imply that $v_{4k} \to v_4$ in

$H_{(1)}(D_0)$. On the other hand, the $v_{4k}$ solve the equation $\operatorname{div}(a_2\nabla v_{4k}) = 0$ near $\overline{D}_0$, so by Lemma 5.7.2 we have the relation (5.7.3) with $v_4$ replaced by $v_{4k}$. As shown above, we can pass to the limit as $k$ goes to $\infty$ and obtain (5.7.3).

We write the relation (5.7.3) as

$$\int_{D_1} a_1^{\bullet}\nabla v_1 \cdot \nabla v_4 - \int_{D_2\setminus B} a_2^{\bullet}\nabla v_1 \cdot \nabla v_4 = \int_{D_2\cap B} a_2^{\bullet}\nabla v_1 \cdot \nabla v_4.$$

We can assume that $|a_2^{\bullet}| > \varepsilon > 0$ on $B$. Taking as $v_1, v_4$ singular solutions to the elliptic differential equations $\operatorname{div}(a_j\nabla v_j) = 0$ with poles located close to the origin and outside $\overline{D}_2$ as in the proofs of Theorems 5.1.1 and 5.1.2, we conclude that the right side is unbounded with respect to these poles, while the left side is bounded. This contradiction shows that $D_1 = D_2$.

The next step is to prove that $a_1^{\bullet} = a_2^{\bullet}$ on $\partial D_1$. As above, let us assume the opposite. Then we can assume that $|a_2^{\bullet}(0) - a_1^{\bullet}(0)| > 0$. We choose a ball $B$ inside $\Omega$ such that $|a_2^{\bullet} - a_1^{\bullet}| > 0$ on $\overline{B}$. Let us replace the coefficient $a_1$ by $a_3$ by extending it from $D_1$ as a $C^1$-function. Repeating the previous approximation argument, from the relation (5.7.3) we obtain

$$\int_{D_1} (a_4 - a_3)\nabla v_3 \cdot \nabla v_4 = 0 \tag{5.7.4}$$

for all solutions $v_j$ to the equations $\operatorname{div}(a_j\nabla v_j) = 0$ near $\overline{D}_1$. Then we can repeat the proof of Theorem 5.1.1 starting with the relation (5.1.1) and conclude that $a_3 = a_4$ and $\nabla a_3 = \nabla a_4$ on $\partial D_1$.

**Lemma 5.7.3**. *Let two conductivity coefficients $a_1, a_2 \in H_{2,\infty}(V)$, where $V$ is a neighborhood of $\partial\Omega$.*

*Then $a_1, a_2$ produce the same Dirichlet-to-Neumann maps if and only if $a_1 = a_2$, $\nabla a_1 = \nabla a_2$ on $\partial\Omega$ and*

$$\int_{\Omega} a_1\nabla u_1 \cdot \nabla u_2 = \int_{\Omega} a_2\nabla u_1 \cdot \nabla u_2$$

*for all solutions $u_j \in H_{(1)}(\Omega)$ to the equations* $\operatorname{div}(a_j\nabla u_j) = 0$ *in* $\Omega$.

PROOF. If $a_1, a_2$ produce the same Dirichlet-to-Neumann maps, then Theorem 5.1.1 claims that they and their gradients are equal on $\partial\Omega$. From (5.0.4) we obtain the orthogonality relations.

If we have the orthogonality relations and $a_1, a_2$ and their gradients are equal on $\partial\Omega$, then formula (5.0.4) gives

$$\int_{\partial\Omega} ((\Lambda_1 - \Lambda_2)u_2)u_1 = 0$$

for all solutions $u_1, u_2$ of the corresponding equations in $H_{(1)}(\Omega)$. Since $u_1$ on $\partial\Omega$ can be any smooth function (say, in $H_{(1/2)}(\partial\Omega)$) we have $(\Lambda_1 - \Lambda_2)u_2 = 0$ again for all smooth $u_2$. So $\Lambda_1 = \Lambda_2$.

The proof is complete. □

From the equalities (5.7.4) and Lemma 5.7.3 we conclude that the coefficients $a_3, a_4 \in H_{2,\infty}(D_1)$ produce equal Dirichlet-to-Neumann maps. By Theorems 5.2.1 and 5.4.1 they are equal. So $a_1 = a_2$ on $D_1$.

The proof is complete.

Uniqueness can be obtained for all piecewise analytic $a_j$ by using the methods described above. Also, we conjecture a logarithmic stability estimate for domains $D_j$ (or $\Omega(k; a_j)$).

Most general results on uniqueness of inclusion $D$ in anisotropic case are given by Kwon [Kw] who detalized the method of singular solutions exposed in the proof of Theorem 5.7.1. The main assumption in [Kw] is that $deta^\bullet \neq 0$ on $\partial D$. Stability of logarithmic type was recently proven by Alessandrini and Di Christo [AC].

**Exercise 5.7.4**. Show that for the equations $\mathrm{div}(a\nabla u) = 0$ and $-\Delta u + cu = 0$ with $a \in H_{2,\infty}(\Omega)$, $c \in L_\infty(\Omega)$ the local Dirichlet-to-Neumann maps uniquely determine $a$ and $c$, provided that these coefficients are known near $\partial\Omega$.

{*Hint*: Let $V$ be a smooth subdomain of $\Omega$ such that $a$ or $c$ is known outside $V$. By using approximation arguments from the proof of Theorem 5.7.1 and Lemma 5.7.3 prove that the local Dirichlet-to-Neumann map on $\Omega$ uniquely determines the complete Dirichlet-to-Neumann map on $V$ and apply known results.}

In so-called resistivity logging in geophysics one considers the local Dirichlet-to-Neumann from the part $\Gamma$ of $\Omega$ which inside $\overline{\Omega}$ and another additional data on $\Gamma$ which are hard to interpret as a complete Dirichlet-to-Neumann map. For some uniqueness results for piecewise constant conductivity from these data we refer to Druskin [Dr2]. In some other applications (like magnetic prospecting) one uses a slightly different set of data, which are generated by exterior sources.

Let us consider the boundary value problem

$$
\begin{aligned}
&-\mathrm{div}(a\nabla u) = \delta(x^*) \text{ in } \mathbb{R}^n;\\
&u(x) \text{ tends to } 0 \text{ as } |x| \text{ tends to } +\infty \text{ when } n \geq 3;\\
&u(x) = C\ln|x| + u_0(x),\\
&\qquad \text{where } u_0 \text{ goes to zero at infinity when } n = 2.
\end{aligned}
\tag{5.7.5}
$$

We assume that $a$ is constant outside a bounded set. Then by using potential theory or the Lax-Phillips device (see Section 6.1) one can prove existence and uniqueness of a solution $u(x, y)$ of the problem (5.7.5), which is actually Green's function of equation (5.7.5) in the whole space. Let $\Omega$ be a domain with analytic boundary that contains this bounded set. When $n = 2$ we assume that $\Omega$ is a half-plane. Let $\Gamma, \Gamma^*$ be two open nonempty subsurfaces of $\partial\Omega$. To identify $a$ we need additional data, and we consider the following:

$$
u(x, x^*) = g(x, x^*) \text{ when } x \in \Gamma, x^* \in \Gamma^*. \tag{5.7.6}
$$

**Exercise 5.7.5**. Prove the uniqueness of $a$ entering the problem (5.7.5) from the data (5.7.6) in one of the following two cases: (1) $a \in C^2(\mathbb{R}^n)$, (2) $a$ satisfies the conditions of Theorem 5.7.1.

{*Hint*: By using that $u(x, y)$ is analytic when $x \neq y \in \partial\Omega$ and symmetric with respect to $x, y$, show that $u(x, y)$ is uniquely determined for all $x, y \in \partial\Omega$ and also (by uniqueness of solutions of exterior Dirichlet problems) for all $x, y$ outside $\Omega$. Obtain the orthogonality relations for solutions $u_1(x, y)$, $u_2(y)$ with $x$ outside $\Omega$. By using the argument form the proof of Lemma 5.7.2 extend these relations onto all solutions $u_1(y)$, $u_2(y)$ of two possible equations in $\Omega$. Apply Lemma 5.7.3 and Theorems 5.2.1, 5.5.1.}

In practical computations one is making use of linearization of inverse problems around the constant $a$. One interesting case is discussed by Engl and Isakov [EnI] in connection with identification of steel reinforcement bars in concrete from measurements of the exterior magnetic field of these bars. In Section 4.5 we somehow justified the linearization when $\operatorname{vol} D$ (and not $\|a^\bullet\|_\infty$) is small and area $\partial D$ is bounded.

Let $u_0$ be a solution to the problem (5.7.5) with $a = 1$ and let $u_1$ be $u - u_0$. Then

$$-\Delta u_1 = \operatorname{div}(a^\bullet \chi(D)\nabla u_0) + \operatorname{div}(a^\bullet \chi(D)\nabla u_1) \text{ in } \mathbb{R}^n$$

and $u_1$ has the same behavior at infinity as $u$. Observe that $u_0(x, y)$ is $-1/2\pi \ln |x - y|$ when $n = 2$, and $1/(4\pi|x - y|)$ when $n = 3$. In Section 4.5 it was shown that (for bounded $\Omega$) $u_1 - u_0$ can be approximately replaced by the solution $v$ to the following problem:

$$-\Delta u = \operatorname{div}(a^\bullet \chi(D)\nabla u_0) \text{ in } \Omega, v = 0 \text{ on } \partial\Omega.$$

A similar argument is valid for the whole $\mathbb{R}^n$, so we replace $u_1 - u_0$ by $v$ satisfying the equation

$$\Delta v = -\operatorname{div}(a^\bullet \chi(D)\nabla u_0 \qquad \text{in } \mathbb{R}^n \tag{5.7.7}$$

and behaving at infinity as $u$ in (5.7.5). The well-known representation of $v$ by potentials gives

$$v(x, x^*) = c_n \int a^\bullet(y)\chi(D)(y)|x^* - y|^{1-n}|x - y|^{1-n}(x^* - y) \cdot (x - y)dy,$$

where $x, x^*$ stand for the positions of the receiver and of the field generator, $c_2 = 1/(2\pi)^2$, $c_3 = 1/(4\pi)^2$.

By letting $x = x^* \in B^*$ (a ball outside $D$), we reduce our original nonlinear inverse problem to the linear integral equation with the Riesz-type kernel

$$\int_\Omega f(y)|x - y|^{2-2n}dy = F(x), \quad x \in B^*, \tag{5.7.8}$$

where $f = c_n a^\bullet \chi(D)$.

## 5.8 Maxwell's and elasticity systems

Recently, uniqueness results for the Schrödinger equation were generalized onto two classical systems of mathematical physics, which was not a simple task because the available proofs for the scalar equations made a substantial use of substitution (5.2.1). This substitution relies on commutativity properties of scalar differential operator that are not valid for most matrix differential operators, in particular for the classical elasticity system. Maxwell's system is not elliptic, so there are additional difficulties in this case.

The stationary electromagnetic field $(E, H)$ of frequency $\omega$ in the medium $\Omega$ of permittivity $\epsilon$, conductivity $\sigma$, and magnetic permeability $\mu$ satisfies Maxwell's equations

$$\text{curl}\,\mathbf{E} = i\omega\mu\mathbf{H}, \quad \text{curl}\,\mathbf{H} = -i\omega(\epsilon + i\sigma/\omega)\mathbf{E} \text{ in } \Omega. \tag{5.8.1}$$

We assume that $\epsilon, \mu$, and $\sigma$ are in $C^3(\Omega) \cap C(\overline{\Omega})$, all of them are nonnegative, and $\epsilon, \mu$ are strictly positive on $\overline{\Omega}$. It is known that with the exception of some discrete set of values of $\omega$ with accumulation point at infinity, for any function $\mathbf{g}_0 \in TH_{(1/2)}(\partial\Omega)$ (the space of tangential vector fields on $\partial\Omega$ with components in $H_{(1/2)}(\partial\Omega)$) there is a unique (weak) solution $(\mathbf{E}, \mathbf{H}) \in L_2(\Omega) \times L_2(\Omega)$ to Maxwell's system with prescribed tangential component $\gamma_\tau\mathbf{E} = \mathbf{g}_0$ on $\partial\Omega$, so we have the well-defined map $\Lambda : \mathbf{g}_0 \to \gamma_\tau\mathbf{H}$ from $TH_{(1/2)}(\partial\Omega)$ into $TH_{(-1/2)}(\partial\Omega)$. We refer for proofs and definitions to the paper of Ola, Päivärinta, and Somersalo [OPS].

**Theorem 5.8.1**. *Assume that a frequency $\omega$ is not in the exceptional set. Assume that $\epsilon, \mu$ are constants $\epsilon_0, \mu_0$ on $\partial\Omega$, and $\sigma$ is zero on $\partial\Omega$.*

*Then the operator $\Lambda$ uniquely determines the coefficients $\epsilon, \sigma$, and $\mu$.*

This result in an important particular case has been obtained by Colton and Päivärinta [CoP] and in the general case by Ola, Päivärinta, and Somersalo [OPS].

Their proofs are quite ingenious. They are based on appropriate orthogonality relations that claim that $\Lambda$ uniquely determines the integrals

$$I = \int_\Omega ((\mu - \mu_0)\mathbf{H} \cdot \mathbf{H}_0 - (\epsilon + i\sigma/\omega - \epsilon_0)\mathbf{E} \cdot \mathbf{E}_0)$$

for all solutions $\mathbf{E}, \mathbf{H}$ to Maxwell's system and all solutions $\mathbf{E}_0, \mathbf{H}_0$ to this system with $\mu = \mu_0, \epsilon = \epsilon_0$, and $\sigma = 0$ with given boundary data. The next step is constructing the special almost exponential solutions

$$\mathbf{E} = e^{i\zeta\cdot}(\zeta(\mathbf{E}) + W(\mathbf{E})), \quad \mathbf{H} = e^{i\zeta\cdot}(\zeta(\mathbf{H}) + W(\mathbf{H}))$$

similar to functions (5.3.3) containing a large parameter ("frequency") $|\zeta|$, which is a delicate part since our system is not elliptic. We assume that

$$\zeta \wedge \zeta(\mathbf{E}) = \omega\mu_0\zeta(\mathbf{H}),\ \zeta \wedge \zeta(\mathbf{H}) = -\omega\epsilon_0\zeta(\mathbf{E}),$$
$$\zeta \cdot \zeta = k^2 \quad \text{with } k = \omega\mu_0\epsilon_0.$$

Then $\mathbf{E}$ and $\mathbf{H}$ with $W(\mathbf{E}) = W(\mathbf{H}) = 0$ solve Maxwell's system with constant parameters $\mu = \mu_0, \ldots$.

Let $G$ be a fundamental solution for the operator $-\Delta - k^2$, $\alpha = \nabla \ln \gamma$, $\beta = \nabla \ln \mu$, $\gamma = \epsilon + i\sigma/\omega$. Then one can show that a solution $(\mathbf{E}, \mathbf{H})$ to the (vector) equation

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = e^{i\zeta\cdot} \begin{pmatrix} \zeta(\mathbf{E}) \\ \zeta(\mathbf{H}) \end{pmatrix} + G \begin{pmatrix} k^2\gamma^\bullet \mathbf{E} + \nabla\alpha \cdot \mathbf{E} + i\omega\mu_0 \nabla \wedge (\mu^\bullet \mathbf{H}) \\ k^2\mu^\bullet \mathbf{H} + \nabla\beta \cdot H - i\omega\epsilon_0 \nabla \wedge (\gamma^\bullet \mathbf{E}) \end{pmatrix}$$

with $\gamma^\bullet = (\gamma - \epsilon_0)/\epsilon_0$, $\mu^\bullet = (\mu - \mu_0)/\mu_0$ is a solution to Maxwell's system. This equation can be written as

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = e^{i\zeta\cdot} \begin{pmatrix} \zeta(\mathbf{E}) \\ \zeta(\mathbf{H}) \end{pmatrix} + G(V+S) \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{5.8.2}$$

where

$$V = \begin{pmatrix} \omega^2(\mu\gamma - \mu_0\epsilon_0) +^t (\nabla\alpha) & i\omega/\gamma\nabla(\mu\gamma)\wedge \\ -i\omega/\mu\nabla(\mu\gamma)\wedge & \omega^2(\mu\gamma - \mu_0\epsilon_0) +^t (\nabla\beta) \end{pmatrix}$$

and

$$S = \begin{pmatrix} \alpha\cdot\nabla & 0 \\ 0 & \beta\cdot\nabla \end{pmatrix}.$$

to solve equation (5.8.2) by contraction arguments as in Section 5.3 it is crucial to use the following matrix identity discovered by Colton and Päivärinta:

$$[\Delta, M] = M(S+Q)$$

with

$$M = \begin{pmatrix} \gamma^{1/2} & 0 \\ 0 & \mu^{1/2} \end{pmatrix}, \quad Q = \begin{pmatrix} \Delta(\gamma^{1/2})/\gamma^{1/2} & 0 \\ 0 & \Delta(\mu^{1/2})/\mu^{1/2} \end{pmatrix}.$$

By using this identity and applying the operators $-(\Delta + k^2)$ and $M$ to (5.8.2), one obtains the following analogue to equation (5.3.11):

$$\begin{aligned} \begin{pmatrix} W(\mathbf{E}) \\ W(\mathbf{H}) \end{pmatrix} &= (M^{-1}M_0 - I) \begin{pmatrix} \zeta(\mathbf{E}) \\ \zeta(\mathbf{H}) \end{pmatrix} \\ &\quad + M^{-1}G_\zeta M(V-Q) \begin{pmatrix} W(\mathbf{E}) \\ W(\mathbf{H}) \end{pmatrix}, \end{aligned} \tag{5.8.3}$$

where $G_\zeta$ is the regular fundamental solution for the operator $-(\Delta + 2i\zeta\cdot + k^2)$, which can be solved by a contraction argument like that in Section 5.3.

Choosing $\zeta = (\tau, i(\tau^2 + R^2)^{1/2}, (R^2 + k^2)^{1/2})$ and

$$\zeta(\mathbf{E}) = (1, 1, -(\zeta_1 + \zeta_2)/\zeta_3), \quad \zeta(\mathbf{H}) = (\omega\mu_0)^{-1}\zeta \wedge \zeta(\mathbf{E}),$$

we satisfy all the conditions above. We define the free space solutions $\mathbf{E}_0$, $\mathbf{H}_0$, replacing $\zeta$, $\zeta(\mathbf{E})$, $\zeta(\mathbf{H})$ in the formulae for $\mathbf{E}$, $\mathbf{H}$ by

$$\zeta^0 = \tau^* - \zeta, \zeta^0(\mathbf{E}) = (1, -1, (\zeta_1 + \zeta_2)/\zeta_3), \zeta^0(\mathbf{H}) = (\omega\mu_0)^{-1}\zeta^0 \Lambda \zeta^0(\mathbf{E})$$

and letting $W = 0$.

Finally, studying the asymptotic behavior of $I$ as the parameter $R$ goes to $\infty$ by means of equation (5.8.3), one obtains the partial differential equation $\Delta u + F(u, v) = pu$. Then interchanging $\zeta(\mathbf{E})$ and $\zeta(\mathbf{H})$ one gets the equation $\Delta v + F(v, u) = qv$. Here $u = (\mu/\mu_0)^{1/2}$ and $v = (\epsilon + i\sigma/k)^{1/2}$, where $F(x, y) = C(1/2 - x^3 y - x^2/2 + x/y + x)$ and $p, q$ are functions determined by the data of the inverse problem. Moreover, $u, v$ are given outside $\Omega$, so by applying uniqueness of continuation for elliptic equations we conclude that they are uniquely determined inside $\Omega$.

This outline can only illuminate the proof but not explain it in sufficient detail given in the original paper [OPS] or in the paper [OS].

For second order systems the situation is more complicated. Eskin [Es1] obtained necessary and sufficient conditions for elliptic matrix equations with the diagonal principal part $-\Delta$:

$$-\Delta\mathbf{u} + \mathbf{B}\nabla\mathbf{u} + \mathbf{C}\mathbf{u} = 0 \text{ in } \Omega \tag{5.8.4}$$

where $\mathbf{u} = (u_1, \ldots, u_m)$, $\mathbf{B} = (\mathbf{B}_1, \ldots, \mathbf{B}_n)$, $\mathbf{B}_j$ are $m \times m$-matrices with entries in $C^\infty(\Omega)$, $\mathbf{C}$ is a $m \times m$ matrix with entries in $L_\infty(\Omega)$. The pair $(\mathbf{B}, \mathbf{C})$ is gauge equivalent to the pair $(\mathbf{B}_\bullet, \mathbf{C}_\bullet)$ if there is a $m \times m$ invertible matrix $\mathbf{G} \in C^\infty(\overline{\Omega})$ such that

$$\mathbf{A}_\bullet = \mathbf{G}^{-1}\mathbf{A}\mathbf{G} + 1/2\mathbf{G}^{-1}\nabla\mathbf{G},$$

$$\sum(1/4\mathbf{B}^2_{j,\bullet} - 1/2\partial_j\mathbf{B}_{j,\bullet}) + \mathbf{C}_\bullet) = \mathbf{G}^{-1}(\sum(1/4\mathbf{B}^2_j - 1/2\partial_j\mathbf{B}_j) + \mathbf{C})\mathbf{G} \text{ on } \Omega$$

where the sums are over $j = 1, \ldots, n$. We will assume that the Dirichlet problem for the system (5.8.4) with the data $\mathbf{u} = \mathbf{g}_0$ on $\partial\Omega$ has a unique solution. It is known that this is true for almost all coefficients and it is not hard to give sufficient conditions by assuming some positivity of matrices $\mathbf{B}, \mathbf{C}$ like in [LU]. Then we have a well-defined Dirichlet-to Neumann map $\Lambda\colon \mathbf{g}_0 \to \partial_\nu\mathbf{u}$ on $\partial\Omega \in Lip$ which is a continuos linear operator from $\mathbf{H}_{(1/2)}(\partial\Omega)$ into $\mathbf{H}_{(-1/2)}(\partial\Omega)$.

**Theorem 5.8.2**. *Let $n \geq 3$. Let $\Omega$ be a ball in $\mathbb{R}^n$. Let matrix-functions* $\mathbf{B}, \mathbf{C}$, $\mathbf{B}_\bullet, \mathbf{C}_\bullet$ *have supports in $\Omega$. Let $\Lambda, \Lambda_\bullet$ be the Dirichlet-to-Neumann operators corrsponding to these sets of coefficients.*

*If $\Lambda = \Lambda_\bullet$, then the the matrix coefficients* $(\mathbf{B}, \mathbf{C})$ *are gauge equivalent to the marix coefficients* $(\mathbf{B}_\bullet, \mathbf{C}_\bullet)$.

Now we consider the stationary elasticity system

$$A^e_i u = \sum \partial_j(c_{ijkl}\varepsilon_{kl}) = 0 \text{ in } \Omega \quad (\text{the sum is over } j, k, l = 1, \ldots, n), \tag{5.8.5}$$

where $\varepsilon_{kl} = 1/2(\partial_l u_k + \partial_k u_l)$ is the linear strain and $c_{ijkl}$ is the elastic tensor with $C^\infty(\overline{\Omega})$-components, they obtained uniqueness at $\partial\Omega$ of this tensor in the case of the classical elasticity ($c_{ijkl} = \lambda\delta_{ik}\delta_{jl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, $\delta_{ij}$ is the Kronecker delta). The natural analogue of the Dirichlet-to-Neumann map $\Lambda_{\lambda,\mu}$ maps the displacement vector $\mathbf{u} = (u_1, \ldots, u_n) = g$ at the boundary to the stress

$$\Lambda_i(\mathbf{g}_0) = \sum \nu_j c_{ijkl} 2^{-1} \varepsilon_{kl} \quad \text{(the sum over } j, k, l = 1, \ldots, n)$$

at the boundary. The boundary reconstruction is considered in [NaU1], [NaU2]. Partial uniqueness results for determination of elastic parameters inside $\Omega$ from all boundary observations were obtained by Eskin and Ralston [ER4] and Nakamura and Uhlmann [NaU3]. In the multidimensional case currently the following result is available.

Let $n \geq 3$. Let the Lame parameters $\lambda, \mu \in C^\infty(\overline{\Omega})$ satisfy the following strong convexity assumption:

$$\mu > 0, n\lambda + 2\mu > 0 \quad \text{on } \Omega.$$

Let $|\nabla \mu| \leq \varepsilon_0$ for some small positive $\varepsilon_0$.

Then the elastic Dirichlet-to-Neumann map $\Lambda_{\lambda,\mu}$ uniquely determines $\lambda$ and $\mu$ in $\Omega$.

Proofs are based on the following orthogonality relations for two possible solutions $\lambda_1, \mu_1, \lambda_2, \mu_2$ of the inverse problem:

$$(5.8.6) \qquad \int_\Omega ((\lambda_1 - \lambda_2) \operatorname{div} \mathbf{u}^1 \cdot \operatorname{div} \mathbf{u}^2 + 2(\mu_1 - \mu_2)(\varepsilon(\mathbf{u}^1) \cdot \varepsilon(\mathbf{u}^2))) = 0$$

for all solutions $u^j$ to the elasticity systems (5.8.5) with $\lambda = \lambda_j$, $\mu = \mu_j$ and further use of almost exponential solutions as in Section 5.3. However, one cannot reduce the classical elasticity system to a diagonal operator with constant coefficients plus zero-order operator and to construct almost complex exponential solutions some smallness assumtions are needed. The first step is to form a fourth-order system by multiplying $A^3$ from the right by the special matrix $A^{e,co}$ (a second-order matrix operator) of "cofactors" of $A^e$ to obtain

$$A = (\mu(\lambda + 2\mu))^{-1} A^e A^{e,co} = \Delta^2 + A_1 \Delta + A_2,$$

where $A_j$ is a (matrix) differential operator of $j$th order. Then one looks for solutions $e^{i\zeta\cdot}V$, so one is using the operators $A_\zeta = e^{-i\zeta\cdot} A e^{i\zeta\cdot}$. By using the pseudodifferential operator $P_\zeta$ with the symbol $(|\xi|^2 + |\zeta|^2)$, we reduce $A_\zeta$ to a second-order pseudodifferential operator

$$A_\zeta^\bullet = \Delta_\zeta^{\bullet 2} + B_\zeta \Delta_\zeta^\bullet + C_\zeta$$

with $\Delta_\zeta^\bullet = P_\zeta^{-1} \Delta_\zeta$ and zero-order pseudodifferential operators $B_\zeta, C_\zeta$.

A crucial step is a diagonalization of this operator by introducing a new extended vector-function $V^* =^t (V, \Delta_\zeta^\bullet V)$ satisfying the first-order system $A_\zeta^* V^* = 0$ with $A_\zeta^* = \Delta_\zeta^\bullet + A^{*0}$, $A^*$ a zero-order $6 \times 6$-matrix differential operator. It turns out that $A_\zeta^*$ is completely diagonalizable:

$$A_\zeta^* A_\zeta^0 = B_\zeta^0 \Delta_\zeta^\bullet \text{ (modulo smoothing operators)}$$

for some zero-order pseudodifferential operators $A_\zeta^0, B_\zeta^0$. Moreover, all these operators are computable when $|\zeta| \to \infty$. Finally, our approximate solutions will be of the form $A^{e,co}(e^{\zeta\cdot}V_\infty(v))$, where $v$ solves a diagonal system with constant coefficients and the operator $V_\infty$ is some standard fundamental solution.

Calculating the limit of the integrals (5.8.6) as $|\zeta| \to \infty$ and using the inverse Fourier transform, one gets second-order hyperbolic equations for $\lambda$ and $\mu$ with respect to space variables. Since the Cauchy data of the Lame parameters on $\partial\Omega$ can be found by boundary reconstruction, these equations uniquely determine $\lambda$ and $\mu$.

A version of the orthogonality relations (5.8.6) can be used to estimate size of an inclusion with different elastic properties as for inverse conductivity problem in section 4.6. We refer to the review paper of Alessandrini, Morassi, and Rosset in [I3].

## 5.9 Open problems

We list some outstanding research problems important for the theory and applications.

**Problem 5.1**. Prove uniqueness of the coefficient $c \in L_\infty(\Omega)$ of the two-dimensional Schrödinger operator $-\Delta + c$.

The available semiglobal uniqueness result (Corollary 5.4.2) follows from the global uniqueness theorem of Nachman for the two-dimensional conductivity equation. Possibilities of this scheme seem to be exhausted, so new ideas are needed.

**Problem 5.2**. Prove completeness of products of solutions for general (elliptic) second-order equations.

This property is known for the equations $L + c$, where $L$ is a partial differential operator with constant coefficients under some additional conditions and for the equations close to these equations. These conditions are satisfied for second-order equations of elliptic, parabolic, and hyperbolic type and for some higher-order equations. We refer to Sections 5.3, 9.5.

**Problem 5.3**. (Local Dirichlet-to-Neumann Map). Let $\Gamma$ be a nonempty open part of $\partial\Omega$. Let $A$ be $-\Delta + c$, $c \in L_\infty(\Omega)$, in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. We define the local Dirichlet-to-Neumann operator $\Lambda(\Gamma)$ that maps the Dirichlet data $g_0 \in H_{(1/2)}(\partial\Omega)$, $g_0 = 0$ on $\partial\Omega\backslash\Gamma$ into the Neumann data on $\Gamma$. Prove that $\Lambda(\Gamma)$ uniquely determines $c$.

When $\Gamma = \partial\Omega$, global uniqueness follows from Theorem 5.2.2 and Corollary 5.5.2. When supp $c \subset \Omega$ (i.e, when $c$ is zero near $\partial\Omega$), one can repeat approximation arguments from Section 5.7 and prove uniqueness, too (see Exercise 5.7.4). Generally, this problem is open even in the three-dimensional case. We think it is quite difficult and important for applications, since it does suggest that boundary measurements can be implemented only on a (arbitrarily small) part of the boundary.

Let us consider the following boundary value problem:

$$-\Delta u + cu = \delta(x^*) \text{ in } \mathbb{R}^3, \quad \lim u(x) = 0 \text{ as } |x| \to \infty,$$

where $c \in L_\infty(\Omega)$, $c = 0$ outside a bounded domain $\Omega$. It solution $u(x; x^*) = 1/(4\pi|x - x^*|) + v(x; x^*)$, where $v$ is continuous when $x, x^*$ are not in $\Omega$.

**Problem 5.4**. Prove uniqueness of $c$ given $v(x; x)$ for all $x \in B^*$, where $B^*$ is a ball $\Omega$ outside $\overline{\Omega}$.

This problem suggests a natural way to reduce overdetermination in the inverse problem with many boundary measurements. While the linearized version possesses the uniqueness property as observed in Sections 2.2, 5.7, we have no idea how to approach this problem with variable $c$ in the original nonlinear case.

**Problem 5.5**. Show that in the three-dimensional case the equality of the Dirichlet-to-Neumann maps corresponding to the two anisotropic elliptic equations $\operatorname{div}(a_j \nabla u_j) = 0$ ($a_j$ are negative symmetric matrices) implies that $a_1$ is related to $a_2$ via the relation (5.4.14), where $\Psi$ is a diffeomorphism of $\overline{\Omega}$ that is identical at $\partial\Omega$.

In the plane case this problem has been solved by Astala,Lassas and Päivärinta [AsLP], Nachman [N3] and Sylvester [Sy1] (see Section 5.5). In the three-dimensional case it is considered by Lassas, Lee, Taylor and Uhlmann under some conditions, the most restrictive of them is that $a_j$ is analytic on $\overline{\Omega}$. The general three-dimensional case is open and seems to be quite difficult, in particular due to the scarcity of conformal mappings in $\mathbb{R}^n$, $n \geq 3$.

**Problem 5.6**. Consider the inverse problem of finding the nonlinear term $c(u)$ of the equation $-\Delta u + c(u) = 0$. Let $u(;\theta)$ be a solution to this equation satisfying the Dirichlet data $u(;\theta) = \theta g$. Assume that $g$ has a positive maximum on $\partial\Omega$ at a point $y$. Show that the Neumann data $\partial_\nu u(y;\theta)$ given for $0 < \theta < \theta^*$ uniquely determine $c$ on $(0, \theta^*)$, provided that $c \in C^2[0, \theta^*]$ and $\partial_u c \geq 0$.

We think that the linearization procedure described in Section 5.6 can be adjusted to this situation. Probably the most difficult part here will be a study of the linearized problem.

**Problem 5.7**. Show that the Dirichlet-to-Neumann map for the quasilinear equation $-\Delta u + b(u, \nabla u) = 0$ determines $b$ on $\mathbb{R}^{n+1}$ in a Hölder (or even Lipschitz) conditionally stable way.

We think that this problem can be solved by using reconstruction at the boundary for linear equations and the linearization technique from Section 5.6. Since both ingredients admit Lipschitz stability estimates in natural and appropriate norms, one can expect the same stability for the resulting reconstruction.

**Problem 5.8**. Assume that we are given the Dirichlet-to-Neumann map for the elliptic equation $\mathrm{div}(a\nabla u) = 0$ in a three-dimensional domain $\Omega$. (i) Prove uniqueness for $D$ entering the conductivity coefficient $a = a_0 + a^\bullet \chi(D)$ without the assumption that $\Omega\backslash\overline{D}$ is connected. (ii) Prove uniqueness of $C^2$-smooth $a_0, a^\bullet$ and of Lipschitz $D$.

This problem looks like a difficult one. It is obviously related to problem 5.3. In the plane case positive answer is given by results of Astala and Päivärinta [AsP]. In the three-dimensional case best regularity assumptions are given by Päiv ärinta, Panchenko, and Uhlmann [PPU].

**Problem 5.9**. Prove uniqueness of discontinuous coefficients $\epsilon = \epsilon_1 + \epsilon^\bullet \chi(D)$, $\mu = \mu_0 + \mu^\bullet \chi(D), \sigma = \sigma_0 + \sigma^\bullet \chi(D)$ of Maxwell's system with given local Dirichlet-to-Neumann map (defined as in Section 5.7) under the assumptions that $\epsilon^\bullet, \mu^\bullet, \sigma^\bullet \in C^3(\overline{D})$, where $D$ is a bounded Lipschitz subdomain of $\Omega$ with connected $\Omega\backslash\overline{D}$, when one of the following conditions is satified: $\epsilon^\bullet \neq 0, \mu^\bullet \neq 0, \sigma^\bullet \neq 0$ on $\partial D$. Prove uniqueness of the discontinuous Lame parameters $\lambda = \lambda_0 + \lambda^\bullet \chi(D), \mu = \mu_0 + \mu^\bullet \chi(D)$ of the classical elasticity system under similar assumptions.

Supposedly, one can adjust the methods of Section 5.7. Apparently, there will be some technical difficulties with constructing singular solutions, but generally this problem looks like a solvable one.

**Problem 5.10**. Show that the elastic Dirichlet-to Neumann map $\Lambda_{\lambda,\mu}$ uniquely determines the Lame parameters $\lambda, \mu \in C^\infty(\overline{\Omega})$ without smallness assumptions.

We mention another very interesting problem on the characterization of the Dirichlet-to-Neumann map, which is to find necessary and sufficient conditions on an operator $\Lambda : H_{(1/2)}(\partial\Omega) \to H_{(-1/2)}(\partial\Omega)$ to be the Dirichlet-to Neumann operator of a second-order (self-adjoint) elliptic equation. This seems to be a very difficult question in part due to instability of the inverse conductivity problem in classical functional spaces and its overdeterminacy.

# 6

# Scattering Problems

## 6.0 Direct Scattering

The stationary incoming wave $u$ of frequency $k$ is a solution to the perturbed Helmholtz equation (scattering by medium)

$$Au - k^2 u = 0 \text{ in } \mathbb{R}^3 \tag{6.0.1}$$

($A$ is the elliptic operator $-\operatorname{div}(a\nabla) + b \cdot \nabla + c$ with $\Re b = 0$, $\operatorname{div} b = 0$, and $\Im c \leq 0$, which coincides with the Laplace operator outside a ball $B$ and which possesses the uniqueness of continuation property) or to the Helmholtz equation (scattering by an obstacle)

$$\Delta u + k^2 u = 0 \text{ in } D_e = \mathbb{R}^3 \backslash \overline{D} \tag{6.0.2}$$

with the Dirichlet boundary data

$$u = 0 \quad \text{on } \partial D \quad \text{(soft obstacle } D\text{).} \tag{6.0.3$_d$}$$

or the Neumann boundary data

$$\partial_\nu u = 0 \quad \text{on } \partial D \text{(hard obstacle } D\text{).} \tag{6.0.3$_n$}$$

The function $u$ is assumed to be the sum of the so-called incident plane wave $u^i(x) = \exp(ik\xi \cdot x)$ and a scattered wave $v$ satisfying the Sommerfeld radiation condition

$$\sigma \cdot \nabla v - ikv(x) = O(r^{-2}) \qquad \text{as } r \text{ goes to } \infty, \tag{6.0.4}$$

where $r = |x|, \sigma = x/r$. Here $\xi \in \mathbb{R}^3$, $|\xi| = 1$, is the so-called incident direction. It is well known (see the book of Colton and Kress [CoKr]) that any solution $u$ to the Helmholtz equation outside $B$ that is the sum of an incident wave and a function $v$ satisfying (6.0.4) admits the representation

$$u(x; \xi, k) = \exp(ik\xi \cdot x) + r^{-1} \exp(ikr)\mathcal{A}(\sigma, \xi, k) + O(r^{-2}). \tag{6.0.5}$$

The function $\mathcal{A}$ is called the *scattering amplitude* (or the *scattering pattern*).

The representation (6.0.5) follows from the fact that any solution $v$ to the Helmholtz equation satisfying the radiation condition (6.0.4) has the representation by a single layer potential

$$v(x) = \int_{\partial B} g(y) K(x - y; k) d\Gamma(y), \tag{6.0.6}$$

where $K(x; k) = e^{ik|x|}/(4\pi|x|)$ and $B$ is some large ball. To obtain (6.0.5) from (6.0.6) it suffices to use the elementary expansion

$$K(r\sigma - y; k) = e^{ikr}/(4\pi r)e^{-ik\sigma\cdot y}(1 + O(r^{-1})),$$

which is uniform with all derivatives with respect to $y \in B$ and bounded $k$.

In this chapter we will assume that $D$ is a bounded domain in $\mathbb{R}^3$ with $C^2$-boundary $\partial D$ and with the connected complement $D_e = \mathbb{R}^3 \backslash \overline{D}$. We expect that all results are valid for bounded open sets with Lipschitz boundaries and connected components of complements and that proofs need only minor modifications thanks to recent progress on elliptic boundary problems in such domains [McL].

First we establish uniqueness of its solution.

**Lemma 6.1.** *If $v$ solves the equation* (6.0.1) *in $\mathbb{R}^3$ and satisfies the radiation condition* (6.0.4)*, then $v = 0$.*

Proof. From the definition (5.0.1) of a weak solution to equation (6.0.1) in $B$ with the test function $\phi = v$ we have

$$\int_{\partial B} \partial_\nu v\overline{v} = \int_B \left(a\nabla v \cdot \nabla\overline{v} + b \cdot \nabla v\overline{v} + (c - k^2)v\overline{v}\right)$$
$$= \int (a\nabla v \cdot \nabla\overline{v} + \overline{b \cdot \nabla v}v + (c - k^2)v\overline{v})$$

when we integrate by parts the term $b \cdot \nabla v\overline{v}$ and use the conditions on $b$. It follows that the integral over $\partial B$ is equal to the sum of two terms: one involving $\nabla v$ and another $cv\overline{v}$. The first term coincides with its complex conjugate, so its imaginary part is zero; and the second one has a nonpositive imaginary part due to the condition on $c$, so

$$\Im \int_{\partial B} \partial_\nu v\overline{v} \leq 0.$$

Since $v$ satisfies the Helmholtz equation and the radiation condition, the known results (e.g., [CoKr], Theorem 2.12) imply that then $v = 0$ outside $B$. By uniqueness of the continuation for the elliptic operator $A - k^2$ we obtain that $v = 0$ in $\mathbb{R}^3$.

The proof is complete.

### The Lax-Phillips method of solving the scattering problem

We assume that $A = -\Delta$ outside a bounded smooth domain $\Omega$. Let $B$ be a ball containing $\overline{\Omega}$. Moreover let $k^2$ be not a Dirichlet eigenvalue for $A$ in $B$. We fix $\Omega_0$

containing $\overline{\Omega}$ with the closure in $B$. Let $\phi$ be a cutoff $C^\infty$-function that is 1 on $\Omega$ and 0 outside $\Omega_0$.

We look for a solution

$$v = w - \phi(w - V) \tag{6.0.6}$$

to equation $Av - k^2 v = f$, where $V(; f^*)$ is a solution to the Dirichlet problem

$$AV - k^2 V = f^* \quad \text{in } B, \qquad V = 0 \text{ on } \partial B,$$

$w$ is a solution to the Helmholtz equation in $\mathbb{R}^3$

$$-\Delta w - k^2 w = f^*$$

satisfying the radiation condition (6.0.4), and $f^*$ is a function to be found later. We have $v = V$ on $\Omega$, so $AV - k^2 V = f$ there. Outside $\Omega$ the operator $A = -\Delta$, and

$$\begin{aligned} Av - k^2 v &= -\Delta w - k^2 w + \Delta\phi(w - V) \\ &\quad +2\nabla\phi \cdot \nabla(w - V) + \phi(\Delta(w - V) + k^2(w - V)) \\ &= f^* + Kf^* + \phi(f^* - f^*), \end{aligned}$$

where we let $Kf^* = \Delta\phi(w - V) + 2\nabla\phi \cdot \nabla(w - V)$. Defining $Kf^*$ as zero in $\Omega$, we conclude that $v$ solves the original equation if and only if $f^*$ solves the following equation:

$$f = f^* + Kf^*. \tag{6.0.7}$$

We claim that the operator $K$ is compact from $H_{p,l}(B)$ into itself. Indeed, the well-known elliptic estimates (Theorem 4.1) give that $V$ and $w$ are linear continuous operators from $H_{l,p}(B)$ into $H_{l+2,p}(B)$. Here $l = -1$ when $a \in L_\infty(B)$, $l = 0$ when $\nabla a \in L_\infty(B)$, and can be greater when the coefficients of $A$ are more regular. Since the definition of $K$ involves only first-order derivatives of $w$, $V$, this operator is continuous from $H_{l,p}(\Omega)$ into $H_{l+1,p}(\Omega)$ and therefore compact into $H_{l,p}(\Omega)$. Finally, the equation (6.0.7) is Fredholm, so its solvability follows from the uniqueness of its solution.

Let $f = 0$. Then $v$ is a solution to the homogeneous scattering problem. By Lemma 6.1 we have $v = 0$ in $\mathbb{R}^3$. So $w = \phi(w - V)$. Observe that $f^* = f = 0$ on $\Omega$ and $V = 0$ on $\Omega$. Then from the equations for $w$ and $V$ we obtain that $w - V$ solves the homogeneous Helmholtz equation in $B$. In addition, $w = 0$, $V = 0$ on $\partial B$. According to the choice of $B$, $k^2$ is not an eigenvalue for $-\Delta$ in $B$. Therefore, $w - V = 0$ on $B$. Hence $w = 0$ on $B$, and from the definition of $w$ we have $f^* = 0$.

It is obvious that this method implies unique solvability of the scattering problem for a general (not necessarily self-adjoint) $A$ provided that we have uniqueness of its solution. At present we do not know how to prove this uniqueness.

**Exercise 6.2.** By using the Lax-Phillips method prove the unique solvability of the obstacle scattering problem and of the problem (5.7.5).

The inverse scattering problem is to find $A$ or $D$ given the scattering amplitude $\mathcal{A}$. This problem is apparently overdetermined ($\mathcal{A}$ is defined on the 5-dimensional manifold $S^2 \times S^2 \times \mathbb{R}_+$), so it is important to study uniqueness when the scattering data are given, say, for one frequency or for one incident direction. However, even the complete knowledge of $\mathcal{A}$ does not guarantee uniqueness of $A$ without additional assumptions.

In Section 6.1 we collect auxiliary results on uniqueness and stability of recovery of a solution $u$ to the Helmholtz equation from its scattering amplitude as well as some lemmas about approximation of an arbitrary solution by particular incident waves, which are of independent interest and which allow us to deduce uniqueness and stability results for the inverse scattering problem from results for finite domains (in particular, from theorems on the Dirichlet-to-Neumann map). In Section 6.2 we give results on inverse scattering by medium, and in Section 6.3 by obstacles.

## 6.1 From $\mathcal{A}$ to near field

Referring to the book of Colton and Kress [CoKr], p. 35, we recall the following known representation

(6.1.1) $$v(x) = k\Sigma h_m^{(1)}(kr) Pr_m \mathcal{A}(\sigma),$$

where $h_m^{(1)}(r)$ is the Hankel function and $Pr_m$ is an orthogonal projector in the space $L_2(S^2)$, $Pr_l Pr_m = 0$ when $l \neq m$, and the sum of these projectors over $m$ is the identity (in fact, it is the spherical harmonics expansion of a function on the unit sphere $S^2$). This representation implies in particular the famous Rellich theorem: the first asymptotic term $\mathcal{A}$ determines a solution $u$ to the Helmholtz equation with regular behavior at infinity (in the above sense). Observe that the Hankel function behaves like $(2m/(er))^m$ for large $m$ uniformly with respect to $r$ on bounded sets.

**Lemma 6.1.1.** *Let*

$$\sum m^{2m} b_m^2 \rho_1^{2m} < M^2 \text{ and } \sum b_m^2 \leq \varepsilon^2.$$

*If* $\varepsilon < 1$, $\rho < 1$, $\rho/\rho_1 < 1/e$, *then*

(6.1.2) $$\sum m^{2m} b_m^2 \rho^{2m} \leq (M^2 + 1)\varepsilon^{2\lambda(\varepsilon)},$$

*where* $\lambda(\varepsilon) = 1/(1 + \ln(-\ln \varepsilon + e))$.

PROOF. Let

$$\sigma = -\ln \varepsilon, N = \sigma/(1 + \ln(\sigma + e)).$$

Then we have

$$\sum_{N \leq m} m^{2m} b_m^2 \rho^{2m} \leq \sum_{N \leq m} m^{2m} b_m^2 \rho_1^{2m} (\rho/\rho_1)^{2m} \leq M^2 e^{-2N} = M^2 \epsilon^{2\lambda(\varepsilon)}.$$

For the first terms we have

$$\sum_{m<N} m^{2m} b_m^2 \rho^{2m} \leq N^{2N}\varepsilon^2 = e^{(2N\ln N - 2\sigma)} \leq e^{(-2\sigma/(1+\ln(\sigma+e))}$$

because $2N\ln N - 2\sigma \leq -2N$. The latter inequality follows from

$$\begin{aligned} N\ln N + N &\leq \sigma(1+\ln(\sigma+e))^{-1}\ln(\sigma(1+\ln(\sigma+e))^{-1}) \\ &\quad + \sigma(1+\ln((\sigma+e))^{-1} \\ &\leq \sigma(1+\ln(\sigma+e))^{-1}\ln\sigma + \sigma(1+\ln(\sigma+e))^{-1} \\ &\leq \sigma \end{aligned}$$

where we used that $\ln(\sigma+e) \geq 0$, $\ln(\sigma+e) \geq \ln\sigma$.

The proof is complete. □

It is not clear that the choice of $N$ in the proof and the estimate (6.1.2) are optimal.

**Lemma 6.1.2.** *Let $v$ be a radiating solution to the Helmholtz equation outside the ball $B_R$, with $|v|_0(\mathbb{R}^3\backslash B_R) \leq C_0$. Let its scattering pattern satisfy the inequality $|\mathcal{A}| < \varepsilon$ on $S^2$.*

*Then*

(6.1.3) $$\|v\|_2(B_{3R+1}\backslash B_{3R}) < C\varepsilon^{\lambda(\varepsilon)},$$

*where $\lambda(\varepsilon) = 1/(1+\ln(-\ln\varepsilon + e))$ and $C$ depends only on $C_0$, $R$.*

PROOF. By using the representation (6.1.1) and the bound on $v$ we conclude that

$$\sum (2m/(ekR))^{2m}\|Pr_m\mathcal{A}\|_2^2(S^2) \leq C.$$

By Lemma 6.1.1 with $b_m = \|Pr_m\mathcal{A}\|$, $\rho_1 = 2/(ekR)$, and $\rho = 2/(ekr) \leq 2/(3ekR)$ we have

$$\|v(r\sigma)\|_2^2(S^2) \leq C\sum (2m/(ekr))^{2m}\|Pr_m\mathcal{A}\|^2 \leq C\varepsilon^{2\lambda(\varepsilon)}$$

when $3R < r$, and we complete the proof by integrating this inequality with respect to $r$ over $(3R, 3R+1)$ and taking the square root. □

Bushuyev [Bus] showed that it is not possible to find $\lambda$ that does not depend on $\varepsilon$, so Hölder stability does not hold. On the other hand, in the same paper he calculated $C$ explicitly and found that for typical computational precision his estimate is very close to the Hölder one (with relatively large $\lambda$). Stability estimates are discussed also by Taylor [Tay].

Now we review a related question about relations between the Dirichlet-to-Neumann map and the scattering amplitude. We fix some ball $B = B_R$.

**Theorem 6.1.3.** *Let $k$ be not an eigenvalue of the Dirichlet problem for $A_j$ in $B$. Let $\mathcal{A}_j(\,;k)$ be the scattering amplitude for the operator $A_j$ in* (6.0.1)*, and let $\Lambda_j$ be the Dirichlet-to-Neumann map for this operator in $B$.*

*Then $\mathcal{A}_1(\,;k) = \mathcal{A}_2(\,;k)$ if and only if $\Lambda_1(\,;k) = \Lambda_2(\,;k)$.*

PROOF. Let $\mathcal{A}_1 = \mathcal{A}_2$. Using the representation (6.1.1) we then obtain $u_1 = u_2$ for scattering solutions outside $B$, and therefore the Neumann data for $u_j$ coincide on $\partial B$. Henceforth $\Lambda_1(u_1) = \Lambda_2(u_1)$. The equality of the Dirichlet-to-Neumann maps on the whole $H_{(1/2)}(\partial B)$ follows from Lemma 6.1.4.

Let $\Lambda_1 = \Lambda_2$. Let $w$ be a solution to the Dirichlet problem $(A_2 - k^2)w = 0$ in $B$, $w = u_1(\,;\xi, k)$ on $\partial B$. Since $u_1$ solves the Dirichlet problem for the operator $A_1 - k^2$ with the same Dirichlet data and this operator has the same Dirichlet-to-Neumann map, we have $\partial_\nu u_1 = \partial_\nu w$ on $\partial B$. Define $w^*$ as $w$ on $B$ and $u_1$ outside $B$. Then $w^*$ is a solution to the scattering problem (6.0.1), (6.0.4) with $A = A_2$. This solution is unique, so $u_2 = w^*$, and therefore $u_1 = u_2$ outside $B$. By comparing the asymptotics (6.0.5) for $u_1$ and $u_2$ we conclude that $\mathcal{A}_1 = \mathcal{A}_2$. The proof is complete. □

**Lemma 6.1.4.** *For any bounded domain $\Omega$ with connected $\mathbb{R}^3\backslash\overline{\Omega}$,* span$\{u(\,;\xi, k)$: $\xi \in \Sigma\}$ *is $L^2(\Omega)$-dense in the space of all solutions to equation* (6.0.1) *near $\overline{\Omega}$.*

PROOF. Let us assume the opposite. Then there is a function $f \in L_2(\mathbb{R}^n)$ supported in $\overline{\Omega}$ such that $\int f\overline{u}(;\xi) = 0$ for all scattering solutions $u(;\xi)$, but not for a solution $u_0$ to equation (6.0.1) near $\overline{\Omega}$. As shown above, there is a solution $w$ to the equation $(A^* - k^2)w = f$ in $\mathbb{R}^3$ satisfying the radiation condition. We pick up a ball $B$ containing $\overline{\Omega}$ where the solution of the Dirichlet problem for $A - k^2$ is unique. By using the definition (4.0.3) of a generalized solution $w$ with the test function $\phi = \overline{u}(;\xi)$, we obtain

$$
\begin{aligned}
0 &= \int_B f\overline{u}(;\xi) \\
&= \int_B \big(a\nabla w \cdot \nabla\overline{u}(;\xi) + b \cdot \nabla w\overline{u}(;\xi) + (\overline{c} - k^2)w\overline{u}(;\xi)\big) - \int_{\partial B} \partial_\nu w\overline{u}(;\xi).
\end{aligned}
$$

Since $u(;\xi)$ solves the homogeneous equation $(A - k^2)u = 0$, we similarly conclude that the right side of the above equation with $w$ and $\overline{u}$ interchanged is zero as well. Finally,

$$\int_{\partial B} (\partial_\nu w\overline{u} - \partial_\nu\overline{u}w) = 0.$$

From the representation (6.0.5) we replace $u$ by the sum of the exponent and a remainder $v$ that satisfies the radiation condition. Since both $w$ and this remainder solve the Helmholtz equation, we can replace the surface of integration $\partial B$ by $\partial B(0; r)$ with $r$ tending to infinity and by using the radiation condition conclude that this integral is zero; so we can replace $u$ by $\exp(ik\xi \cdot x)$. Indeed, from the representation (6.0.6) it follows that for radiating solutions $w, v$ one has $|w| + |v| \le Cr^{-1}$, $|\nabla w| + |\nabla v| \le Cr^{-2}$.

Let $w_0$ be a solution to the Helmholtz equation in $B$ with the Dirichlet data $w = w_0$ on $\partial B$. Since $u_\xi = \exp(ik\xi \cdot x)$ satisfies the same equation in $B$, again from Green's formula the integral of $\partial_\nu w_0\overline{u}_\xi - w_0\partial_\nu\overline{u}_\xi$ is zero. Comparing the two integrals, we have that the integral of $(\partial_\nu w_0 - \partial_\nu w)\overline{u}_\xi$ over $\partial B$ is zero. Since

$k^2$ is not an eigenvalue of $-\Delta$, one can conclude ([Is5]) that exponential solutions are dense in $L_2(\partial B)$, so $\partial_\nu w_0 = \partial_\nu w$ on $\partial B$. Let us define $w^*$ as $w_0$ on $B$ and $w$ outside $B$. Since $w, w_0$ have the same Cauchy data on $\partial B$, the function $w^*$ solves the Helmholtz equation on $\mathbb{R}^3$ and satisfies the radiation condition. By Lemma 6.1, $w^* = 0$ on $\mathbb{R}^3$, so $w = 0$ outside $B$ and on $B$ by uniqueness of the continuation from outside $\Omega$.

Now, consider $\int_B f\overline{u}_0$. Using that $(A^* - k^2)w = f$ and $w = 0$ outside $\Omega$, from Green's formula we obtain that the integral of $f\overline{u}_0$ is zero, which is a contradiction. □

**Corollary 6.1.5.** *Let $\Omega$ be any bounded domain with analytic boundary and with connected $\mathbb{R}^3\setminus\overline{\Omega}$ and let $A = -\Delta$ outside $\Omega$ and near $\partial\Omega$. Let $k^2$ be not a Dirichlet eigenvalue for $A$ in $\Omega$.*

*Then* $\operatorname{span}\{u(;\xi)\}$ *is dense in $H_{(1/2)}(\partial\Omega)$.*

To prove this result we first observe that from Lemma 6.1.4 and interior Schauder-type estimates it follows that solutions $u(;\xi)$ are $H_{(1)}$-dense on bounded subsets among all solutions. Since $k^2$ is not an eigenvalue, one can solve the Dirichlet problem in $\Omega$ with any real-analytic Dirichlet data. This solution can be continued into some neighborhood of $\overline{\Omega}$ and therefore can be $H_{(1)}(\Omega)$-approximated by solutions $u(;\xi)$. By trace theorems we have also an approximation in $H_{(1/2)}(\partial\Omega)$. Since real analytic functions on $\partial\Omega$ are dense among all functions, we complete the proof.

One has an analogy of these approximation results for scattering on obstacles.

**Lemma 6.1.6.** *For any bounded domain $\Omega$ with $\overline{\Omega} \subset \mathbb{R}^3\setminus\overline{D}$,* $\operatorname{span}\{u(,\xi;k) : \xi \in \Sigma\}$ *is $L_2(\Omega)$-dense in the space of all solutions to the obstacle problem* (6.0.2), (6.0.3$_d$), (6.0.3$_d$) *(or* (6.0.3$_n$)*) in $V$, which is an open subset of $\mathbb{R}^3\setminus\overline{D}$ such that $\overline{\Omega} \subset V$ and $\mathbb{R}^3\setminus(\overline{D \cup V})$ is connected.*

PROOF. We will slightly modify the proof of Lemma 6.1.4. As in that proof, we find $f$ in $L_2(\Omega)$ orthogonal to all radiating solutions to the, say, hard obstacle problem, but not to a solution of this problem in $V$. According to Exercise 6.2 there is a solution $w$ to the equation $-(\Delta + k^2)w = f$ in $\mathbb{R}^3\setminus\overline{D}$ satisfying the boundary condition (6.0.3$_n$) and the radiation condition. Let $B$ be a ball containing $\overline{D \cup \Omega}$ where the solution to the Dirichlet problem for the Helmholtz equation is unique. We have

$$0 = \int_{B\setminus D} f\overline{u}(\,,\xi) = \int_{B\setminus D} \left(\nabla w \cdot \overline{\nabla u}(,\xi) - k^2 w\overline{u}(,\xi)\right) - \int_{\partial B} \partial_\nu w\overline{u}(,\xi),$$

where we integrated by parts and used the zero Neumann condition for $w$ on $\partial D$. If we integrate by parts once more and use that $u$ solves the Helmholtz equation, we conclude that

$$\int_{\partial B} (w\,\partial_\nu\overline{u}(,\xi) - \partial_\nu w\overline{u}(,\xi)) = 0.$$

Arguing as in Lemma 6.1.4, we can replace $u$ by $\exp(ik\xi\cdot)$ and conclude that $w = 0$ outside a (large) ball $B$. Now,

$$\int_\Omega f\overline{u}_0 = \int_V (-\Delta - k^2)w\overline{u}_0 = \int_{\partial V} (\partial_\nu \overline{u}_0 w - \partial_\nu w \overline{u}_0) = 0$$

because at the points $\partial V \cap \partial D$ the normal derivatives of $w$ and $\overline{u}_0$ are zero, while at the remaining points of $\partial V$ we have $w = \partial_\nu w = 0$ by uniqueness of the continuation from $B$. We have a contradiction, which shows that the initial assumption was wrong.

The proof is complete. □

## 6.2 Scattering by a medium

From the results of Chapter 5 and Section 6.1 we can derive many important results about uniqueness and stability in inverse scattering.

**Theorem 6.2.1.** *Let $A_j = -\Delta + c_j$, where $c_j \in L_\infty(B)$ (or $A_j = -\operatorname{div}(a_j\nabla)$, where $a_j \in H_{2,\infty}(B)$).*

*If $\mathcal{A}_1(\sigma, \xi, k) = \mathcal{A}_2(\sigma, \xi, k)$ for all $\sigma \in S^2$, $\xi \in S^2$, and for one $k$, then $c_1 = c_2$ (or $a_1 = a_2$).*

This result follows from Theorems 5.2.1, 5.2.2, and 6.1.3.

The remarks of section 5.2 and theorems of Section 5.5 similarly lead to uniqueness results under the assumptions that $c_j \in L_p(B)$ or $a_j \in H_{2,p}(B)$, where $p \geq n/2$ if $n \geq 3$ and $a_j \in L_\infty(B)$ if $n = 2$.

We have to mention that this result was obtained by Nachman [N1] (who based it on the Sylvester-Uhlmann theorem) and by Novikov [No1], where another fruitful viewpoint is adopted.

Hähner and Hohage [HaH] obtained the following conditional logarithmic stability estimate.

**Theorem 6.2.2.** *Let $c_1, c_2 \in H_{(2)}(B)$ for some ball $B$ and $c_1 = c_2 = 0$ outside $B$. Let $\|c_j\|_{(2)}(B) \leq M$. Let $\mathcal{A}_j$ be the scattering pattern of the operator $-\Delta - k^2 + c$.*

*Then there is a constant $C$ depending only on $B, k, M$ such that*

$$\|c_2 - c_1\|_2(B) \leq C(-ln\||\mathcal{A}_2(,k) - \mathcal{A}_1(,k)\|_2(S^2 \times S^2))^{1/4}$$

They used some ideas of Alessandrini of proving stability from the Dirichlet-to Neumann exposed in section 5.2 and properties of Green functions and scattering patterns as well as stability of recovery of near field from $\mathcal{A}$ in part described in section 6.1. This stability estimate looks optimal due to recent results of Mandache [Ma] and Di Christo and Rondi [DR] and it indicates that inverse scattering by a medium is a strongly ill-posed problem.

By using two different frequencies one is able to recover two coefficients of $A$.

**Corollary 6.2.3.** *Let $A_j = -\operatorname{div}(a_j\nabla) + c_j$ where $a_j$ and $c_j$ satisfy the conditions of Theorem 6.2.1.*

*If $\mathcal{A}_1(\sigma, \xi, k) = \mathcal{A}_2(\sigma, \xi, k)$ for all $\sigma \in S^2$, $\xi \in S^2$, and two different $k$, then $a_1 = a_2$ and $c_1 = c_2$.*

To prove this corollary we make use of the substitution (5.2.1), which does not change the equations and their solutions outside $B$ and which replaces the equations inside $B$ by

$$-\Delta u^* + (a_j^{-1/2}\Delta a_j^{1/2} + (c_j - k^2)a_j^{-1/2})u^* = 0.$$

By Theorem 6.2.1 we conclude that the coefficients of this equation are equal for two different $k$ when $j = 1, 2$, so $a_2 = a_1$ and hence $c_2 = c_1$.

Similar conclusions are valid for identification of several coefficients of elliptic equations when we combine Theorems 5.4.1 and 6.1.3.

Let $\partial\Omega \in C^2$, $\mathbb{R}^3\backslash\overline{\Omega}$ be connected, and $\operatorname{supp} c \subset \Omega$. Now we focus for a while on the Schrödinger operator $A = -\Delta + c$. Writing (6.0.1) as the equation $-\Delta v - k^2 v = -cu$ in $\mathbb{R}^3$, using the classical integral representation of scattering solutions of such equations, and letting $u = \exp(ik\xi\cdot) + v$ we obtain the so-called Lippman-Schwinger integral equation

$$u(x;\xi) = \exp(ik\xi \cdot x) - \int K(x-y;k)c(y)u(y;\xi)\,dy. \tag{6.2.1}$$

When $c$ is compactly supported, it is quite easy to prove unique solvability of this equation and therefore of the original scattering problem (6.0.1), (6.0.4). On the other hand, a simple generalization of Green's representation implies that

$$v(x;\xi) = \int_{\partial\Omega} (v(;\xi)\partial_\nu K(x-;k) - \partial_\nu v K(x-;k)) \text{ when } x \in \mathbb{R}^3\backslash\overline{\Omega}.$$

By Green's formula the right side of this equation with $v$ replaced by $\exp(ik\xi\cdot)$ is zero, since this exponential function solves the Helmholtz equation inside $\Omega$. Therefore, we can replace $v$ by $u - \exp(ik\xi\cdot)$ on the left side and by $u$ on the right side. Using the jump relations for double layer potentials and the equality $\partial_\nu u = \Lambda u$ on $\partial\Omega$, we arrive at the relation

(6.2.2)

$$u(x;\xi) = \exp(ik\xi \cdot x) + (u(x;\xi)/2 + B_k u(;\xi)(x) - S_k\Lambda u(;\xi)(x)),\ x \in \partial\Omega,$$

where

$$B_k g(x) = \int_{\partial\Omega} \partial_{\nu(y)} K(x-y;k)g(y)d\Gamma(y)$$

is the double layer potential of density $g$ on $\partial\Omega$ and

$$S_k g(x) = \int_{\partial\Omega} K(x-y;k)g(y)d\Gamma(y)$$

is the single layer potential. It can be shown that the operator $I/2 + B_k - S_k\Lambda$ is compact (in $H_{(s)}(\partial\Omega)$, $0 < s < 3/2$), and a solution $u(;\xi)$ to equation (6.2.2) is unique; so the Dirichlet-to-Neumann map uniquely determines scattering solutions $u(;\xi)$ on $\partial\Omega$. An asymptotic analysis of both sides of the formula for $v$ when $|x| \to \infty$ shows that

$$\mathcal{A}(\sigma, \xi, k) = -1/(4\pi) \int_{\partial\Omega} e^{-ik\sigma\cdot y}(\Lambda + ik\sigma \cdot \nu(y))u(y, \xi)d\Gamma(y). \tag{6.2.3}$$

For details and a proof we refer to the paper of Nachman [N1]. From (6.0.5) and (6.2.1) by letting $|x| \to +\infty$ we obtain

$$\mathcal{A}(\sigma, \xi, k) = -1/(4\pi) \int e^{-ik\sigma\cdot y} c(y)u(y;\xi)dy. \tag{6.2.4}$$

For small $\|c\|_\infty$ we conclude from (6.2.1) that $u(x;\xi) \sim \exp(ik\xi \cdot x)$ on $\Omega$, so the following (Born) approximation makes sense and is widely used:

$$\mathcal{A}(\sigma, \xi; k) \sim -1/(4\pi) \int e^{-ik(\sigma-\xi)\cdot y} c(y)\, dy. \tag{6.2.5}$$

We explained above how to obtain a uniqueness result for inverse scattering from results for the Dirichlet-to-Neumann map. In fact, the opposite direction is quite useful. In the paper of Nachman [N1] there is a constructive algorithm to recover $c$ from the Dirichlet-to-Neumann map $\Lambda$ for equation (6.0.1) with $A = -\Delta + c$ at given frequency $k$. The complex scattering solution $\psi(x, \zeta)$ is defined as a solution of the Schrödinger equation $(-\Delta + c)\psi = 0$ in $\mathbb{R}^3$, where $\psi(x, \zeta) = \exp(ix \cdot \zeta) + w(x, \zeta)$ and $w \in L^2_{-\delta}(\mathbb{R}^n)$ with $1/2 < \delta < 1$. Here $L_{\delta,2}$ is the weighted $L_2$-space with the norm $\|(1 + |x|^2)^\delta v\|_2(\mathbb{R}^n)$. The existence of such almost exponential solutions for large $|\zeta|$, $\zeta \cdot \zeta = 0$, has been proved by Sylvester and Uhlmann [SyU2], who also showed that $\|w\|_2(\Omega) < C/|\zeta|$. Similarly to the case of real $\zeta$, one can show that these complex scattering solutions satisfy the equation

$$\psi(, \zeta) = \exp(ix \cdot \zeta) + (I/2 + B_\zeta - S_\zeta\Lambda)\psi(, \zeta), \tag{6.2.6}$$

where $S_\zeta$ and $B_\zeta$ are classical single and double layer potentials that correspond to the Faddeev Green's function (the inverse Fourier transform of the function $\xi^2 + 2\zeta \cdot \xi$ multiplied by $\exp(ix \cdot \zeta)$). It is proven by Nachman [N1] that for large $|\zeta|$, $\zeta \cdot \zeta = 0$, the integral equation (6.2.6) has a unique solution $\psi \in H_{(3/2)}(\partial\Omega)$. Define an analogue of the scattering amplitude divided by $\exp(ix \cdot \zeta)$,

$$\begin{aligned} t(\xi, \zeta) &= \int_\Omega e^{-ix\cdot(\xi+\zeta)} c(x)\psi(x, \zeta)dx \\ &= \int_{\partial\Omega} \exp(-ix \cdot (\xi + \zeta))(\Lambda + i(\xi + \zeta) \cdot \nu))\psi(, \zeta)d\sigma(x), \end{aligned}$$

where we have used the equality $c\psi = \Delta\psi$, Green's formula, the fact that $\exp(-ix \cdot (\xi + \zeta))$ solves the Laplace equation with respect to $x$ when $\xi \cdot \xi + 2\xi \cdot \zeta = 0$, and that $\partial_\nu\psi = \Lambda\psi$. The function $t(\xi, \zeta)$ is known due to the last representation as an integral over $\partial\Omega$. As mentioned above, $\psi(x, \zeta)\exp(-ix \cdot \zeta)$ is convergent to 1 in $L_2(\Omega)$ as $|\zeta| \to \infty$. Hence the Fourier transformation of $c$ at a point $\xi$ is equal to $\lim t(\xi, \zeta)$ as $|\zeta|$ goes to $\infty$ and $\xi^2 + 2\zeta \cdot \xi = 0$, and we have a constructive method of finding $c$. As shown in Section 5.3 (proof of Corollary 5.3.5), one can find such $\zeta$. It is possible to avoid a limiting procedure and to recover $c$ by using an integral representation of Koppelman type where singular integrals over the manifold $\{\zeta : \xi^2 + 2\zeta \cdot \xi = 0\}$ and some Bochner-Martinelli kernels are involved ([N1], Theorem 3.4).

Similar ideas and interesting results (including also a characterization of the data in the inverse scattering) are contained in the papers of Henkin and Novikov [HeN] and Novikov [No1], who were inspired by the pioneering work of Faddeev in the 1960s where almost exponential solutions with complex $\zeta$ were first introduced and used. There are space limitations that do not allow us to go into the details of these papers. We only mention that in [HeN] there is a characterization of the scattering amplitude in terms of solutions of some complicated integral equations and compatibility conditions for the $\overline{\partial}$-equation on a certain complex-analytic manifold where solutions of these equations are defined. There is a possibility that these results can be simplified and applied to the characterization of the Dirichlet-to-Neumann map, which is a completely open problem in the three-dimensional case. To related questions we refer also to the book of R. Newton [Ne] and to the papers of Nachman [N1], [N3].

The difficulties with existence and stability in the inverse scattering problem with complete data stimulated some formulations with restricted scattering amplitude. Probably the most reasonable and popular are data $\mathcal{A}(\sigma, -\sigma; k)$ (the inverse backscattering). While global uniqueness is still open, there is one recent result by Eskin and Ralston [ER2] showing stability near "almost all" potentials $c$.

Let $H_{\alpha,N}$ be the weighted Hölder space that is obtained by completion of the space $C_0^\infty(\mathbb{R}^3)$-functions $\hat{c}$ with respect to the norm

$$\|\hat{c}\|_{\alpha,N} = |(1 + |\xi|^2)^{N/2}\hat{c}(\xi)|_\alpha(\mathbb{R}^3),$$

where $0 < \alpha < 1$, $1 < N$. This space was introduced by Faddeev and Friedrichs. Let $H^r_{\alpha,N}$ consist of Fourier transformations of real-valued potentials.

Let $\mathcal{A}^*(\xi)$ be the backscattering amplitude $\mathcal{A}(|\xi|^{-1}\xi, -|\xi|^{-1}\xi, |\xi|)$. Eskin and Ralston [ER2] obtained the following result.

**Theorem 6.2.4.** *There is an open dense subset $\mathcal{U}$ of $H_{\alpha,N}$ such that its intersection with $H^r_{\alpha,R}$ and the backscattering map $S : \hat{c} \to \mathcal{A}^*$ is an analytic homeomorphism in a neighborhood of any $\hat{c} \in \mathcal{U}$.*

The proof of this result is quite long and technical, so we are able to tell only a few words about it. First, one proves that the map $S$ is differentiable on an open dense subset of $H_{\alpha,N}$ that is a complex Banach space, so we get analyticity of this map. The next step is to show that the Fréchet derivative of $S$ is a Fredholm operator of zero index, which one might expect since in view of the formula (6.2.4) at the origin ($c = 0$) this derivative is the identity. This goal is achieved by representing this derivative as the identity plus a compact operator and plus another operator with compact square. A final argument is application of Fredholm analytic theory. The underlying reason is that the operator of multiplication by $c$ is an analytic operator in an appropriate Banach space, and probably the most difficult part of the proof is to find the right space to secure compactness of related operators.

We emphasize that Theorem 6.2.4 guarantees that the inverse backscattering problem is (locally) stable. However, still there are questions about local invertiblity for real-valued potentials $c$ and about global uniqueness that are somehow similar to those questions about the two-dimensional Dirichlet-to-Neumann map.

The two-dimensional case of inverse backscattering is considered by Eskin and Ralston [ER3] as well.

Now assume that we are given the complete scattering data. Even in this case one cannot determine an operator $A$, and we must impose additional assumptions. Indeed, the substitution $u = c^{\phi} v$ with $\phi = 0$ outside some ball $B$ discussed in Section 5.4 does not change scattering data while changing the differential equation.

**Theorem 6.2.5.** *Suppose that* $A_j = -\Delta + b_j \cdot \nabla + c_j$, *where* $b_j \in C_0^2(\mathbb{R}^n)$, $c_j \in L_\infty(\mathbb{R}^n)$, *are compactly supported, and* $\Re b_j = 0$, $\operatorname{div} b_j = 0$, $\Im c_j \leq 0$.

*If* $\mathcal{A}_1(\sigma, \xi; k) = \mathcal{A}_2(\sigma, \xi; k)$ *for all* $\sigma, \xi \in S^2$ *and all* $k > 0$, *then* $\operatorname{curl} b_1 = \operatorname{curl} b_2$ *and* $4c_1 + b_1 \cdot b_1 = 4c_2 + b_2 \cdot b_2$.

At a fixed frequency $k$ this result under the additional regularity assumption that $b_j \in C^6$, $c_j \in C^5$ follows from the papers of Eskin and Ralston and of Eskin [Es1] and for infinitely smooth coefficients from the paper of Nakamura, Sun, and Uhlmann [NSU] combined with Theorem 6.1.3. Proofs of this result and its relation to hyperbolic problems are given by Novikov and Henkin [HeN] under some additional assumptions. About the variable principal part we refer to the chapter on hyperbolic equations. One can prove that the complete scattering data (i.e., the scattering amplitude given at all frequencies) uniquely determines the lateral hyperbolic Dirichlet-to-Neumann operator. Indeed, by Theorem 6.1.3 the scattering amplitude uniquely determines the elliptic Dirichlet-to-Neumann map for equation (6.0.1) for all real $k$. By using the Fourier or the Laplace transformations it is possible to show that the hyperbolic map is also uniquely determined. So Theorem 6.2.5 follows from Theorem 8.3.1.

Scattering for nonlinear equations was considered by Morawetz [Mo]. Our Theorem 5.6.1 implies some uniqueness results for such equations from scattering data that are obtained in the paper of Isakov and Nachman [IsN].

## 6.3 Scattering by obstacles

First we consider scattering by obstacles $D \subset B(0; R)$. Let $N = \sum(2n+1)$ over all $n$ such that $0 < t(n, l) < k^* R$, where $t(n, l)$ is a zero of the Bessel function $j_n$. Observe that $N = 0$ when $k^* R < \pi$ ([CoKr], p. 107). Let $k_* \leq k_1 < \cdots < k_{N+1} \leq k^*$.

**Theorem 6.3.1** (Uniqueness of Soft Obstacles). *Let us consider two soft scatterers $D_1$, $D_2$ contained in $B$. Let their scattering amplitudes $\mathcal{A}_1$, $\mathcal{A}_2$ coincide either (a) for all $\sigma \in S^2$, one $\xi \in S^2$, and $N+1$ frequencies $k_1, \dots, k_{N+1}$ or (b) for all $\sigma \in S^2$, $N+1$ distinct $\xi$, and one $k \le k^*$.*

*Then $D_1 = D_2$.*

We give a proof in case (a) referring for case (b) to the book of Colton and Kress ([CoKr], section 5.1) and to the paper of Kirsch and Kress [KirK], where they corrected the initial proof of Schiffer described in the book of Lax and Phillips [LaxP2].

Let $D^\infty$ be the unbounded component of the complement $\mathbb{R}^3 \backslash (\overline{D}_1 \cup \overline{D}_2)$. By the conditions of Theorem 6.3.1 we have the same scattering amplitudes for both domains, so from the Rellich uniqueness theorem we obtain that the related solutions $u_1, u_2$ are equal on $D^\infty$. Let $D_\infty$ be $\mathbb{R}^3 \backslash \overline{D^\infty}$. Let us assume that $D_1 \ne D_2$. Then there is a point, say, in $D_2 \backslash \overline{D}_1$. Since the complement of $\overline{D}_1$ is connected, we can join this point with a point in $D^\infty$ by a continuous path that does not intersect $\overline{D}_1$. This path intersects $\partial D^\infty$, so there is a connected component $D_*$ of $D_\infty \backslash \overline{D}_1$.

The function $u_1$ solves the Helmholtz equation in $D_*$ for $k = k_1, \dots, k_{N+1}$. Moreover, $u_1 = 0$ on $\partial D_*$ because this boundary condition is satisfied on $\partial D_1$ and because $u_1 = u_2 = 0$ on $\partial D_2 \cap D^\infty$. According to the Courant max-min property, the eigenvalues of the Laplacian for $D_*$ are greater than corresponding eigenvalues for the ball $B$, which can be calculated explicitly using Bessel functions. In particular, their number is less than $N$. Therefore, one of $k_1^2, \dots, k_{N+1}^2$ is not an eigenvalue. For this $k^2$ we have $u_1 = 0$ on $D_*$. Then $u_1 = 0$ on $D^\infty$ by uniqueness of the continuation. This is a contradiction because $u_1 \ne 0$ outside of a large ball due to the representation (6.0.5). This contradiction shows that $D_1 = D_2$.

The gap in Schiffer's initial proof is due to the possible disconnectedness of $\mathbb{R}^3 \backslash (\overline{D}_1 \cup \overline{D}_2)$.

Observe that in the paper of Stefanov [St] there is a discussion of a moving obstacle and a counterexample that shows nonuniqueness in the inverse scattering problem.

In Theorem 6.3.2 we let $k_j = k_* + (j-1)(k_* - k^*)/N$.

**Theorem 6.3.2** (Stability for Soft Obstacles). *Consider two star-shaped domains $D_1$, $D_2$ given in polar coordinates by the equations $\{r < d_j(\sigma)\}$, where $1/R_0 < |d_j|_{2+\lambda}(S^2) < R_0$.*

*If $|\mathcal{A}_1 - \mathcal{A}_2|(\sigma, \xi, k) < \varepsilon$ for some $\xi \in S^2$, all $\sigma \in S^2$, and $k = k_1, \dots, k_{N+1}$, then $|d_1 - d_2| < C \ln(-\ln \varepsilon)^{-1/C}$, where $C$ depends only on $R_0$.*

A proof of this estimate is given in the papers [Is8], [Is9]. It is based on Lemma 6.1.2, and then on estimates for the continuation of solutions to the Helmholtz equation $(u_2 - u_1)$ from outside of $B$ up to $\partial(\overline{D}_1 \cup \overline{D}_2)$ and from $D_*$ back to outside of $B$ (with bounds explicitly involving some size of $D_*$) where one can always assume that $|u_2| > 1/2$. Combining all explicit bounds, we arrive at the final estimate.

We observe that for analytic boundaries this weak estimate can be improved: as shown in the papers of the author [Is8], [Is9], one can replace the right side of the stability estimate by $C|\lambda(\varepsilon)\ln\varepsilon|^{-1/C}$, where $\lambda(\varepsilon)$ is as defined in Lemma 6.1.2.

We will outline a proof of a stability estimate, referring for detail to the above mentioned papers. In this outline $C$ denotes (different) constants determined only by $R_0$.

The first claim is that

$$|u_j|_2(\mathbb{R}^3\backslash D_j) \le C. \tag{6.3.1}$$

This basically follows from Theorem 4.1 combined with behavior at infinity. The estimate (6.3.1) in view of the well-known integral representation

$$v_j(x) = \int_{D_j} (-\Delta - k^2)v_j^* K(x-)$$

(6.0.5) and of decay of $K$ at infinity implies that

$$|u_j| > 1/2 \quad \text{on } \partial B, \tag{6.3.2}$$

where $B$ is some ball determined only by $R_0$. Here $v_j^*$ is an extension of $v_j$ onto $\mathbb{R}^3$ whose $|\ |_2$-norm is bounded by the $C^2$-norm of $v_j$ (or equivalently of $u_j$) on $\mathbb{R}^3\backslash D_j$. Such $v_j^*$ exists by extension theorems.

Lemma 6.1.2 applied to $v = v_1 - v_2$ gives

$$|u_2 - u_1|_0(B(0;R_2)\backslash B) < \varepsilon_1, \text{ where } \varepsilon_1 = C\varepsilon^{\lambda(\varepsilon)} \tag{6.3.3}$$

and $\lambda(\varepsilon)$ is as defined in Lemma 6.1.2. The next step is to derive from this estimate the following estimate up to the boundary of the analyticity domain of $u_j$

$$|u_2 - u_1|_0(\partial(D_1 \cup D_2)) < \varepsilon_2 = |\ln\varepsilon_1|^{-1/C}. \tag{6.3.4}$$

The estimate (6.3.4) is similar to the logarithmic estimate of Exercise 3.1.2 and is typical for the Cauchy problem for elliptic equations.

Since $u_1 = 0$ on $\partial D_1$ and $|u_1| \le |u_1 - u_2| + |u_2| \le \varepsilon_2$ on $\partial D_2\backslash D_1$, where $u_2 = 0$ by using the Dirichlet problem at one of the frequencies $k_j$ that is not an eigenvalue, we conclude that

$$|u_1| < C\varepsilon_2 \text{ on } D_2\backslash D_1. \tag{6.3.5}$$

The decisive step of the proof is the estimate of the continuation of $u_1$ from $D_2\backslash D_1$ to $\partial B$ based on the methods of complex analysis. To demonstrate this step we introduce the cone $\{x : \||x|^{-1}x - e| < 1/C,\ |x| < \rho\}$ and denote by $\text{con}(x^0; e, \rho)$ its translation by $x^0$. According to our conditions, there is $C$ such that $\text{con}(x^0;\rho)$ defined as $\text{con}(x^0; |x^0|^{-1}x^0, \rho)$ is contained in $\mathbb{R}^3\backslash D_1$ when $x^0 \in \partial D_1$. We claim that

$$\text{if } |u_1| < \varepsilon_3 \text{ on } \text{con}(x^0;\rho),\ \text{then } |u_1| < C\varepsilon_3^{\rho^C/C} \text{ on } \partial B \cap \text{con}(x^0;R). \tag{6.3.6}$$

To prove (6.3.6) we consider $x \epsilon\ \text{con}(x^0;R)$ and make use of polar coordinates $r, \sigma$ in $\mathbb{R}^3$ centered at $x^0$, which is the intersection of the ray $\{tx, t > 0\}$ and $\partial D_1$. We take $\{tx\}$ as the $x_1$-axis in these new coordinates. Then $x = (r, 0, 0)$. Points of $\partial D_1$ will be written as $(y_1, y_2, y_3)$. The function $u_1$ has a complex-analytic continuation onto a sector $Cr_2^2 < r_1^2$ of the complex plane $r = r_1 + ir_2$. To show this, we make use of the above integral representation of $v_1$. Since $\partial D_1$ is the graph of a Lipschitz function in the original polar coordinates, we find $C_1$ such that the cone $\{0 < y_1, C_1^2(y_2^2 + y_3^2) < y_1^2\}$ is outside $D_1$. We have

$$\begin{aligned}|(x-y)\cdot(x-y)| &\geq \Re(x-y)\cdot(x-y)\\ &= r_1^2 - 2y_1r_1 + y_1^2 + y_2^2 + y_3^2 - r_2^2\\ &\geq (r_1^2 - 2y_1r_1 + (1 + 1/(2C_1^2))y_1^2) + 1/2(y_2^2 + y_3^2) - r_2^2\\ &\geq 1/2(y_2^2 + y_3^2)\end{aligned}$$

when $Cr_2^2 < r_1^2$ because by elementary inequalities the three terms in parentheses are greater than

$$(1 - \frac{2C_1^2}{2C_1^2 + 1})r_1^2$$

so these three terms $-r_2^2$ are greater than $1/(2C_1^2 + 1)r_1^2 - r_2^2$, which is nonnegative when we choose $C > 2C_1^2 + 1$. Hence the fundamental solution $K$ has a complex-analytic continuation onto the sector $Cr_2^2 < r_1^2$, which has absolute value bounded by a function integrable over $\partial D_1$. Therefore, $v_1$ and consequently $u_1$ have the above-mentioned continuations bounded by $C$.

Now we can apply methods of complex variables. $u_1$ considered as a function of $r$ is bounded in the sector $\{Cr_2^2 < r_1^2, 0 < r_1 < C\}$ and is less than $\varepsilon_3$ on the segment $[0, \rho]$. As in Section 3.3 we have $|u_1| \leq C\varepsilon_3^{\mu(r)}$, where $\mu(r)$ is the harmonic measure of the interval $[0, \rho]$ in this sector at a point $r$. By using conformal mappings and scaling, one can show that $(\rho/r)^C \leq C\mu(r)$ (see [Is4], p. 88, or [Is8]). Putting all this together, we obtain the claim (6.3.6).

The end of the proof of Theorem 6.3.2 is short. Due to continuity we can assume that $\|d_2 - d_1\|_\infty = (d_2 - d_1)(\sigma_0)$ at some $\sigma_0 \in S^2$. Let $\rho$ be one-half of this number. Then $\text{con}(d_1(\sigma_0)\sigma_0; \rho) \subset D_2\backslash D_1$. From (6.3.3), (6.3.4), (6.3.5), and (6.3.6) we have

$$|u_1| < C(-\lambda(\varepsilon)\ln\varepsilon)^{-\rho^C/C} \text{ at a point of } \partial B.$$

Using the condition (6.3.2) we conclude that

$$1 < C(-\lambda(\varepsilon)\ln\varepsilon))^{-\rho^C/C},$$

and taking logarithms,

$$-C < -\rho^C \ln(-\lambda(\varepsilon)\ln\varepsilon) \text{ or } \rho < C\ln(-\lambda(\varepsilon)\ln\varepsilon)^{-1/C}.$$

Since $C\lambda(\varepsilon) > (-ln\varepsilon)^{-1/2}$ we can drop $\lambda(\varepsilon)$ and remembering that $\rho = 1/2\|d_2 - d_1\|_\infty$ we complete this outline of a proof of Theorem 6.3.2.

We observe that De Christo and Rondi [DR] showed optimality of conditional logarithmic type stability estimate given by Theorem 6.3.2.

Due to the remark at the beginning of Section 6.2, one can use only one frequency $k \le k^* < \pi R$. There is a well-known conjecture that one frequency is sufficient even without this restriction, but there is no idea how to attack it.

In case of the Neumann boundary condition we know weaker results.

**Theorem 6.3.3** (Uniqueness for Hard Obstacles). *Let $D_1$, $D_2$ be two domains with connected $\mathbb{R}^3\backslash\overline{D}_j$ and let $\mathcal{A}_1$, $\mathcal{A}_2$ be their scattering amplitudes corresponding to the Neumann boundary conditions on $\partial D_1$, $\partial D_2$.*

*If $\mathcal{A}_1 = \mathcal{A}_2$ at all $\sigma \in S^2$, $\xi \in S^2$ and one $k$, then $D_1 = D_2$.*

PROOF. Let $D^\infty$ be the unbounded connected component $\mathbb{R}^3\backslash(\overline{D}_1 \cup \overline{D}_2)$, $D_\infty$ the complement of its closure, and $V$ an open set (with smooth boundary) containing $\overline{D}_\infty$. From Green's formula for any solution $v$ to the Helmholtz equation near $\overline{V}$ we obtain

$$\int_{\partial D_1} u_1 \partial_\nu v = \int_{\partial V} (\partial_\nu u_1 v - u_1 \partial_\nu v).$$

Since the scattering data coincide, we conclude as above that $u_1 = u_2$ on $\mathbb{R}^3\backslash V$, and so

$$\int_{\partial D_2} u_2 \partial_\nu v = \int_{\partial D_1} u_1 \partial_\nu v$$

for all such $v$.

If obstacles do not coincide, then as in the proof of Theorem 6.3.1 we can assume that there is a point $x^0 \in \partial D_2\backslash\overline{D}_1$ that belongs also to $\partial D^\infty$. We can let $x^0 = 0$ and assume that the $x_3$-axis coincides with the exterior normal to $\partial D_2$ at 0. Let $\Gamma = \partial D_2 \cap B(0;\delta)$ for some small $\delta$. Since $u_2(;\xi) = u_1(;\xi)$ on $D^\infty$, we can replace $u_2$ on $\Gamma$ by $u_1$ in the preceding integrals and obtain

$$\int_\Gamma u_1(\,;\xi)\partial_\nu v = -\int_{\partial D_2\backslash\Gamma} u_2(;\xi)\partial_\nu v + \int_{\partial D_1} u_1(;\xi)\partial_\nu v. \tag{6.3.7}$$

Let us introduce $x(\varepsilon) = (0, 0, \varepsilon)$, $0 < \varepsilon < \delta/2$. We can find a large ball $B$ such that the mixed boundary value problem for the Helmholtz equation in $B\backslash\overline{D}_1$ and $B\backslash\overline{D}_2$ with the Dirichlet data on $\partial B$ and the Neumann data on $\partial D_j$ is uniquely solvable. In particular, there is a unique solution $u_{1\varepsilon}$ to the boundary value problem

$$(-\Delta - k^2)u_1 = \delta(-x(\varepsilon)) \quad \text{in } B\backslash\overline{D}_1,\ \partial_\nu u_1 = 0 \text{ on } \partial D_1,\ u_1 = 0 \text{ on } \partial B.$$

By letting $u_{1\varepsilon} = K(x(\varepsilon), ) + w$, where $K$ is the (outgoing) fundamental solution to the Helmholtz equation (given after (6.0.6)), and using Theorem 4.1 as well as the assumption about unique solvability of the boundary value problem one can conclude that $\|w\|_\infty(B\backslash D_1) + \|\nabla w\|_\infty(B\backslash D_1) < C$, which does not depend on $\varepsilon$. Using Lemma 6.1.6 and the argument to prove Corollary 6.1.5, we can approximate $u_{1\varepsilon}$ by radiating solutions $u_1(;\xi)$ in $H_{(1)}(\Omega)$, where $\Omega$ is a bounded smooth subdomain of $B\backslash\overline{D}_1$ containing $\overline{D}_\infty\backslash(\overline{D_1 \cup B(0;\varepsilon)})$ and coinciding in $B(0;\varepsilon)$ with $D_2$. We can choose $\Omega$ to be independent on $\varepsilon$. In particular, we can replace $u_1(;\xi)$ on the left side of (6.3.7) by $u_{1\varepsilon}$, and the second integral on the right side of this equality will be bounded with respect to $\varepsilon$ when $v = K(x(\varepsilon), )$. To show that the first term also remains bounded, we recall that $u_2(;\xi) = u_1(;\xi)$ on $B \backslash D_\infty$, and since $u_1(;\xi)$ approximates $u_{1\varepsilon}$ in $H_{(1)}(\Omega)$, we can derive by using local estimates of Theorem 4.1 that the solution $u_2(;\xi)$ to the Helmholtz equation with $\partial_\nu u_2 = 0$ on $\partial D_2$ is bounded in $L_2(\partial D_2 \backslash \Gamma)$. Putting all this together, we obtain from (6.3.1) that

$$\int_\Gamma u_{1\varepsilon} \partial_\nu K(x(\varepsilon), ) = \text{ bounded function of } \varepsilon.$$

Letting $D_{2\delta} = D_2 \cap B(0;\delta)$ and using that $K = K(x(\varepsilon), )$ solves the Helmholtz equation after integrating by parts we will have

$$\int_{D_{2\delta}} \nabla u_{1\varepsilon} \cdot \nabla K = \int_\Gamma u_{1\varepsilon} \partial_\nu K + \int_{\partial D_{2\delta} \backslash \Gamma} u_{1\varepsilon} \partial_\nu K + k^2 \int_{D_{2\delta}} u_{1\varepsilon} K,$$

and since the second and the third term of the right side are bounded with respect to $\varepsilon$, we conclude that

$$\int_{D_{2\delta}} \nabla u_{1\varepsilon} \cdot \nabla K = \text{ bounded function of } \varepsilon. \tag{6.3.2}$$

By using the explicit formula for $K$ and the decomposition $u_{1\varepsilon} = K + w$ it is not difficult to see that

$$|\nabla u_{1\varepsilon}(y) \cdot \nabla K(x(\varepsilon), y)| \geq 1/C|x(\varepsilon) - y)|^{-4},$$

so as in the proof of Theorem 5.1.1, the left side of (6.3.2) is unbounded when $\varepsilon \to 0$.

We have obtained a contradiction, showing that $D_1 = D_2$. □

A proof of Theorem 6.3.3 was given first by Kirsch and Kress [KirK]. It is a modification of the original uniqueness proof for penetrable obstacles in [Is5] that we adopted above to hard obstacles. Recently, Colton and Kirsch [CoK] used singular solutions in a numerical method for the inverse scattering problem.

We observe that there are no uniqueness proofs for hard obstacles when scattering data are given only at one $\xi$ and at a finite number of frequencies $k$. Schiffer's idea cannot be used here, because it is generally wrong that the space of solutions of the homogeneous Neumann problem in the domain $D_*$ (whose boundary might have arbitrary cusps) is finite-dimensional.

To conclude, we consider penetrable obstacles $D$. In this case $u$ solves the equation $(A - k^2)u = 0$, where $A = \mathrm{div}(a\nabla)$ with piecewise analytic $a$ of with $a = 1 + (k-1)\chi(D)$, where $k \in C^2(B)$. In the last case we can write this equation in a more traditional form by letting $u_e = u$ on $D_e$ and $u_i = u$ on $D$. Then this equation with the discontinuous coefficient $a$ is equivalent to the relations

$$\Delta u_e + k^2 u_e = 0 \text{ in } D_e, \quad \mathrm{div}(k\nabla u_i) = 0 \text{ in } D,$$
$$u_e = u_i, \qquad \partial_\nu u_e = k\partial_\nu u_i \text{ on } \partial D,$$

where $u_e \in C^1(\overline{D}_e)$ and satisfies the radiation condition and $u_i \in C^1(\overline{D})$ as soon as $\partial D \in C^2$.

Let us consider two problems with coefficients $a_1, a_2$ and scattering amplitudes $\mathcal{A}_1, \mathcal{A}_2$.

**Theorem 6.3.4** (Uniqueness of Penetrable Obstacles). *Let $a_1, a_2$ satisfy one of the following three conditions:*

(1) *They are piecewise analytic in $\Omega = B$ and their analyticity domains $\Omega(k; a_j)$ have piecewise analytic boundaries.* (2) *They are constant on $\Omega(k; a_j)$.* (3) $a_j = 1 + (k_j - 1)\chi(D_j)$, *where $k_j \in C^2(B)$, $k_j \neq 1$ on $\partial D_j$, and the $D_j$ are unknown open subsets of $B$ with Lipschitz boundaries and connected $\mathbb{R}^3 \setminus \overline{D}_j$.*

*If $\mathcal{A}_1 = \mathcal{A}_2$ at all $\sigma \in S^2$, all $\xi \in S^2$, and one $k$, then $a_1 = a_2$.*

As above, this theorem follows from a simple modification of Theorem 5.7.1 ($A$ is replaced by $A - k^2$) and Theorem 6.1.3

In particular, $D_1 = D_2$ and $k_1 = k_2$ on $D_1$. In the plane case we can conclude only that the domains coincide and $k_1 = k_2$ on $\partial D_1$. By using frequencies one can identify more terms as above. We refer to [Is5].

We think that it is possible to obtain (logarithmic-type) stability estimates similar to those of Theorem 6.3.2, but it will require a modification of technique and new ideas. This remark is also valid for hard obstacles. More general transmission conditions at $\partial D$ are considered by Valdivia [V].

Starting with the paper of Keller of 1958, there are results about stable recovery of (the convex hull of) $D$ from the high-frequency behavior of the kernel $S(\sigma, \xi; k)$ of the scattering operator in the translation representation, which can be found from the scattering amplitude $\mathcal{A}$.

Lax and Phillips [LaxP1] proved that the support function of $D$ is given by the formula $h_D(\xi) = \sup x \cdot \xi$ over $x \in D$ can be recovered from $S$ by the following formula: the right endpoint of the support of $S(\sigma, \xi; s)$ with respect to $s$ is equal to $h_D(\sigma - \xi)$ in the case of soft obstacles in the general case and for hard obstacles when $\sigma = -\xi$ (backscattering). In the paper of Majda and Taylor [MT] there is a uniqueness result for transparent (penetrable) convex $D$ with the given backscattering data. In fact, they first recover Gaussian curvature from the high-frequency behavior $\mathcal{A}$ and then make use of the Minkowski problem.

An interesting (and still) open question concerns uniqueness in the penetrable case when $\mathcal{A}$ is given for one $\xi$ and for several $k$. It has a direct relation to inverse hyperbolic problems with unknown discontinuity surface of speed of propagation, which we will discuss later.

In the paper of Hettlich [Het] the Fréchet derivative with respect to a hard or transparent obstacle is found. It could be useful in the theoretical or numerical analysis of inverse scattering.

## 6.4 Open problems

**Problem 6.1.** Prove uniqueness of scalar $a$, $\operatorname{curl} b$, and $b \cdot b - 2 \operatorname{div} b + 4c$ when one is given the scattering amplitude $\mathcal{A}(\sigma; \xi, k)$ for all $\sigma, \xi \in S^2$, $k = k_1, \ldots, k_N$, where $N$ depends only on the norms $|a|_3(\Omega)$, $|b|_2(\Omega)$, and $|c|_1(\Omega)$ and diam $\Omega$. Show that $N$ can be chosen smaller than $CM$, where $M$ is the maximum of these norms.

This problem is a further development of the results of the paper of Isakov and Sun [IsSu2] on global uniqueness of the potential of the two-dimensional Schrödinger operator when $\mathcal{A}$ is given at finitely many frequencies. On the other hand, it suggests a realistic improvement of Theorem 6.2.5. Very often, one can implement measurements of physical fields at finitely many frequencies quite easily.

**Problem 6.2.** Prove uniqueness of the coefficient $c \in L_\infty(\Omega)$ of the Schrödinger operator $-\Delta + c$ with the given backscattering data $\mathcal{A}(\sigma; -\sigma, k)$ for all $\sigma \in S^2$ and $k \in \mathbb{R}$.

Local (and generic) uniqueness and stability results in this problem have been obtained by Eskin and Ralston (Theorem 6.2.3). Global uniqueness of singularities of $c$ was proven by Greenleaf and Uhlmann [GrU]. At present there is no decisive idea how to get global uniqueness.

**Problem 6.3.** Prove uniqueness of a soft obstacle $D$ when its scattering amplitude $\mathcal{A}(\sigma; \xi, k)$ is given at arbitrary fixed $k$ and $\xi$ for all directions $\sigma \in S_0$. Here $S_0$ is a nonempty open part of the unit sphere.

This is a well-known question that supposedly can be solved by elementary means. However, it has been open for thirty to forty years. For polygonal obstacles there are recent positive results by Cheng and Yamamoto [CheY3].

**Problem 6.4.** Prove uniqueness of a hard obstacle $D$ when $\mathcal{A}(\sigma; \xi, k)$ is known at some (probably small) fixed $k$, fixed $\xi$, and all $\sigma \in S_0$.

Schiffer's method cannot be applied here because contrary to the case of soft obstacles, one cannot control the space of solutions to the homogeneous Neumann problem in a component of the difference of two possible obstacles. Since this component can have arbitrary cusps of its boundary, one cannot even conclude that this space is finite-dimensional. It is of interest of course to find a hard obstacle and the coefficient of the boundary condition simulteneously and from the scattering data at fixed frequency and incident direction. The study of this problem started by Kress and Rundell [KR2].

**Problem 6.5.** Prove uniqueness of a soft obstacle $D$ with $|\mathcal{A}(, \xi; k)|$ on $S^2$.

This problem is motivated by applications. Some local results in the plane case are given by Kress and Rundell [KR1].

# 7

# Integral Geometry and Tomography

The problems of integral geometry are to determine a function given (weighted) integrals of this function over a "rich" family of manifolds. These problems are of importance in medical applications (tomography), and they are quite useful for dealing with inverse problems in hyperbolic differential equations (integrals of unknown coefficients over ellipsoids or lines can be obtained from the first terms of the asymptotic expansion of rapidly oscillating solutions and information about first arrival times of a wave). There has been significant progress in the classical Radon problem when manifolds are hyperplanes and the weight function is unity, there are interesting results in the plane case when family of curves is regular (resembling locally family of straight lines) or in case of family of straight lines with arbitrary regular attenuation. Still there are many interesting open questions about the problem with local data and simulteneous recovery of density of a source and of attenuation. We give a brief review of this area, referring for more information to the book of Natterer [Nat].

## 7.1 The Radon transform and its inverse

We describe here some uniqueness and stability results of recovery of a function given its integrals over a family of hyperplanes (straight lines in $\mathbb{R}^2$ or planes in $\mathbb{R}^3$).

The Radon transform $Rf$ of a compactly supported function $f \in C(\mathbb{R}^2)$ is defined as the integral

(7.1.1) $$F(\omega, s) = Rf(\omega, s) = \int_{x\cdot\omega=s} f.$$

When $f \in L_1(\mathbb{R}^n)$ and compactly supported, the Radon transform is defined by the same formula for almost all $(\omega, s) \in \Sigma \times \mathbb{R}$. Due to the rotational invariancy of $Rf$, we can make use of the expansion of $f$, $F$ in spherical harmonics:

(7.1.2) $$f(x) = \sum f_{j,m}(r)Y_{j,m}(\sigma), \quad F(\omega, s) = \sum F_{j,m}(s)Y_{j,m}(\omega).$$

We recall that a spherical harmonic $Y_{j,m}(\sigma)$ is a homogeneous harmonic polynomial of degree $j$ restricted to the unit sphere. There are $M(j,n)$ ($M(j,2) = 2$, $M(j,3) = 2j+1$) such linearly independent polynomials. The functions $Y_{j,m}$ and $Y_{k,l}$ with different $j$ and $k$ are $L_2$-orthogonal on the unit sphere. By applying the standard orthogonalization procedure we can assume that $Y_{j,m}$ and $Y_{j,l}$ are orthogonal when $m \neq l$ as well, and that the whole system is orthonormal in $L_2(S^{n-1})$.

The following result is due to Cormak ($n = 2$) [Cor] and Deans [D].

**Theorem 7.1.1.** *We have*

$$f_{j,m}(r) = c_n r^{2-n} \int_r^{+\infty} (s^2 - r^2)^{(n-3)/2} C_j^{(n-2)/2}(s/r) F_{j,m}^{(n-1)}(s)ds, \tag{7.1.3}$$

*where $C_j^k(t)$ is a normalized Gegenbauer polynomial of degree $j$, $c_2 = -1/\pi$, $c_3 = \pi^{-3/2}/2$.*

To prove the theorem we need the Funk-Hecke formula:

$$\int_{S^{n-1}} h(\sigma \cdot \omega) Y_{j,m}(\theta) dS(\sigma) = c(n) \int_{-1}^{1} h(t) C_j^{(n-2)/2}(t)(1-t^2)^{(n-3)/2} dt Y_{j,m}(\omega)$$

(7.1.3)

relating spherical harmonics and Gegenbauer polynomials. Here $c(2) = 2$, $c(3) = 2\pi$. Detailed references on this classical result and further properties of Gegenbauer polynomials related to the multiplicative analogue of the Fourier transformation can be found in the book of Natterer [Nat]. Another needed tool is the so-called Mellin transform

$$Mf(s) = \int_0^\infty f(r) r^{s-1} dr,$$

which plays the same role for the multiplicative convolution

$$f *_m g(s) = \int_0^\infty f(r) g(s/r) r^{-1} dr$$

as does the Fourier transformation for convolutions. Here $f, g$ are bounded measurable functions with compact supports. We will need the following known properties of the Mellin transform:

$$\begin{aligned} M(f *_m g)(s) &= Mf(s) Mg(s), \\ Mf(s+t) &= M(x^t f(x))(s), \\ Mf^{(k)}(s) &= (1-s)\dots(k-s) Mf(s-k). \end{aligned} \tag{7.1.4}$$

We observe that by introducing new variables $\xi = \ln r$, $\eta = \ln s$ we replace the Mellin transform by the Laplace transform and the multiplicative convolution by the usual one, so properties (7.1.4) are not completely surprising.

**Exercise 7.1.2.** Prove properties (7.1.4).

Also, we will use in the proof the following known results about the Mellin transform of Gegenbauer polynomials. Let $g(r)$ be $C_j^{(n-2)/2}(r)(1-r^2)^{(n-3)/2}$ when $0 \leq r \leq 1$ and 0 when $1 < r$, and let $g_1(r)$ be $C_j^{(n-2)/2}(1/r)(1-r^2)^{(n-3)/2}$ when $0 \leq r \leq 1$ and 0 when $1 < r$. Then

$$Mg(s) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma(s)2^{-s}}{\left(\Gamma\left(\frac{j+s+n-1}{2}\right)\right)\Gamma\left(\frac{s+1-j}{2}\right)},$$

$$Mg_1(s) = \frac{2^{s-1}\Gamma(n-2)\Gamma\left(\frac{s-j}{2}\right)\Gamma\left(\frac{n-2+s+j}{2}\right)}{\left(\Gamma\left(\frac{n-2}{2}\right)\Gamma(s+n-2)\right)}, \tag{7.1.5}$$

provided that $j < s$. Here $\Gamma(s)$ is Euler's gamma function.

PROOF OF THEOREM 7.1.1. First, we will parametrize the hyperplane $\{x \cdot \omega = s\}$ by points $\sigma$ of the unit sphere $\Sigma$ according to the formula $\sigma \to (s/(\sigma \cdot \omega))\sigma$ and write the integral over the hyperplane as the integral over $\Sigma$,

$$\int_{x\cdot\omega=s} f_{j,m}(r)Y_{j,m}(\sigma)d\Gamma(x) = \int_\Sigma f_{j,m}(s/(\sigma\cdot\omega))Y_{j,m}(\sigma)s^{n-1}/(\sigma\cdot\omega)^n d\Sigma(\sigma).$$

Using the Funk-Hecke theorem (7.1.3) with $h(t) = f_{j,m}(s/t)s^{n-1}/t^n$ when $t > 0$ and 0 otherwise, we obtain

$$\begin{aligned}
&\int_{x\cdot\omega=s} f_{j,m}(r)Y_{j,m}(\sigma)d\Gamma(x)\\
&= c_n \int_0^1 f_{j,m}(s/t)s^{n-1}/t^n C_J^{(n-2)/2}(t)(1-t^2)^{(n-3)/2}dt Y_{j,m}(\omega)\\
&= c_n \int_s^\infty C_j^{(n-2)/2}(s/r)(1-s^2/r^2)_{(n-3)/2} f_{j,m}(r)r^{n-2}dr Y_{j,m}(\omega),
\end{aligned}$$

where we have made use of the substitution $r = s/t$. By comparing with (7.1.2) we conclude that

$$F_{j,m}(s) = c_n \int_s^\infty C_j^{(n-2)/2}(s/r)(1-s^2/r^2)^{(n-3)/2} f_{j,m}(r)r^{n-2}dr. \tag{7.1.6}$$

The equality (7.1.6) can be considered as an Abel-type integral equation with respect to $f_{j,m}$, which can be solved explicitly using the Mellin transform. Letting $f(r) = f_{j,m}(r)r^{n-1}$ and defining $g$ as above, we conclude that $MF_{j,m}(s) = Mf(s)Mg(s)$ because of the first property (7.1.4). To make use of the second formula (7.1.5) we have to replace $s$ by $s-1$. So by applying the second property (7.1.4) with $t = -1$ we will have

$$M(r^{-1}f)(s) = M(f)(s-1)\frac{MF_{j,m}(s-1)}{Mg(s-1)}.$$

Comparing the two formula (7.1.5) we conclude that

$$\frac{1}{Mg(s-1)} = d_n \frac{\Gamma(s+n-2)}{\Gamma(s-1)} Mg_1(s),$$
$$d_n = \frac{\Gamma(n-2)}{\Gamma((n-1)/2)\Gamma((n-2)/2)\Gamma(1/2)},$$

($d_2 = 1/2$). By using the second and third properties (7.1.4) with $t = n-2$ and $k = n-1$ we obtain

$$M(r^{-1}f)(s) = (-1)^{n-1} d_n M(r^{n-2} F_{j,m}^{(n-1)})(s) M(g_1)(s).$$

Returning to the functions from their Mellin transforms by using the first property (7.1.4), we finally will have

$$r^{n-2} f_{j,m} = (-1)^{n-1} d_n g_1 *_m (r^{n-1} F_{j,m}^{(n-1)}),$$

which is formula (7.1.3) by the definitions of the multiplicative convolution $*_m$ and the function $g_1$.

This completes the proof. □

We recall that the (normalized) Gegenbauer polynomial $C_j^k(t)$ of degree $j$ can be defined as

(7.1.7) $$\Gamma^{-1}(k) \sum (-1)^m \Gamma(j-m+k)/(m!(j-2m)!)(2t)^{j-2m}$$

(the sum is over $m < j/2$). When $k = 0$ the Gegenbauer polynomial is the Chebyshev polynomial $T_j(t) = \cos(j \arccos t)$, and when $k = 1/2$ it is the Legendre polynomial $P_j$. By using properties of these polynomials we will obtain later stability estimates for inversion of the Radon transform in the situation of the following corollary.

**Corollary 7.1.3** (The "hole" Theorem). *Let $K$ be a convex compact set in $\mathbb{R}^n$. Let $f \in L_1(\mathbb{R}^n)$ with compact support.*

*If $Rf(\omega, s) = 0$ for (almost) all hyperplanes $\{x \cdot \omega = s\}$ not intersecting $K$, then $f = 0$ outside $K$.*

PROOF. Formula (7.1.6) says that $F(\omega, s)$ given for all $s > R$ uniquely determines $f(r\sigma)$ when $r > R$. In other words, this corollary is true when $K$ is the ball $B(0; R)$. By using scaling and translations we obtain the result also for any ball $B$.

In the general case let $x$ be outside $K$. Then, due to the convexity of $K$ one can find a ball $B$ containing $K$ and not containing $x$. By applying the result for $B$ we get $f(x) = 0$.

The proof is complete. □

For the inversion of the complete Radon transform one can give sharp stability estimates. We introduce the norm

$$\|F\|_{\bullet,\alpha} = \left( \int_{|\sigma|=1} \int_{\mathbb{R}} (1+\tau^2)^\alpha |F(\sigma,\tau)|^2 d\tau dS(\sigma) \right)^{1/2},$$

where $\hat{F}(\sigma, \tau)$ is the (partial) Fourier transform of the function $F(\sigma, s)$ with respect to $s$.

The next sharp stability result was obtained by Natterer ($n = 2$) and by Smith, Solmon, and Wagner [SmSW] in the general case.

**Theorem 7.1.4.** *There is a constant $C = C(n, \alpha, \rho)$ such that*

$$C^{-1}\|f\|_{(\alpha)} \leq \|Rf\|_{\bullet,\alpha+(n-1)/2} \leq C\|f\|_{(\alpha)} \tag{7.1.8}$$

*for all functions $f$ in $C_0^\infty(B(0;\rho))$.*

PROOF. Observe that

$$\hat{F}(\sigma, \tau) = \int_{\mathbb{R}} e^{-i\tau s}\left(\int_{x\cdot\sigma=s} f(x)d\Gamma(x)\right)ds = \int_{\mathbb{R}} e^{-i\tau x\cdot\sigma} f(x)dx = \hat{f}(\tau\sigma).$$

Therefore,

$$\begin{aligned}\|F\|^2_{\bullet,\alpha+(n-1)/2} &= c_n \int_{|\sigma|=1}\int_{\mathbb{R}} (1+\tau^2)^{\alpha+(n-1)/2}|\hat{f}(\tau\sigma)|^2 d\tau d\Gamma(\sigma)\\ &= c_n \int (1+|\xi|^2)^{\alpha+(n-1)/2}|\xi|^{-n+1}|\hat{f}(\xi)|^2 d\xi.\end{aligned} \tag{7.1.9}$$

To obtain the last equality we let $\xi = \tau\sigma$ and used that $d\xi = c_n\tau^{n-1}d\tau d\Gamma(\sigma)$.

Since $(1+|\xi|^2)^{1/2}|\xi|^{-1} \geq 1$, we have the inequality

$$\|F\|^2_{\bullet,\alpha+(n-1)/2} \geq c_n \int (1+|\xi|^2)^{\alpha}|\hat{f}(\xi)|^2 d\xi.$$

The last integral is $\|f\|^2_{(\alpha)}$. Therefore, we have the first inequality (7.1.8).

To prove the second we will split the integration domain in the last integral in (7.1.9) into $|\xi| \geq 1$ and $|\xi| < 1$. In the first integral

$$(1+|\xi|^2)^{(n-1)/2}|\xi|^{-n+1} \leq 2^{(n-1)/2},$$

so this integral is bounded by $C\|f\|_{(\alpha)}$. The second integral is bounded by $C\|\hat{f}\|^2_\infty$. Using that $f$ is compactly supported and applying the Hölder inequality, one has

$$\|\hat{f}\|_\infty \leq C\|f\|_2(B(0;\rho)) \leq C\|f\|_{(\alpha)}.$$

the proof is complete. □

**Theorem 7.1.5.** *Let $n = 2$ or 3. Assume that $|f|_{n-1}(B_R) \leq M$. Let $A_{R,\rho} = \{\rho < s < R\}$.*

*Then there is a constant $C = C(R, \rho)$ such that*

$$\|f\|_2(B_R\backslash B_\rho) \leq CM^2/|\ln\|F\|_2(A_{R,\rho})|.$$

PROOF. As a preliminary step we obtain a bound on Gegenbauer polynomials by using (7.1.7) with $k = 0 (n = 2)$ of $k = 1/2 (n = 3)$. By the basic property of the gamma function,

$$\Gamma(j-m+1/2) = (j-m-1/2)\dots 3/2 1/2\Gamma(1/2) \le (j-m)!\Gamma(1/2).$$

therefore,

$$|C_j^k(x)| \le \sum_{2m<j} (j-m)!/(m!(j-2m)!)(2x)^{j-2m} \le$$

$$\sum_{2m<j} j!/((2m)!(j-2m)!)(2x)^{j-2m} \le (1+2x)^j \tag{7.1.10}$$

according to the binomial theorem.

First we will bound the integral

$$\int_\rho^R f_{j,m}^2 r^{n-1} dr.$$

We will start with the three-dimensional case, which is easier technically. According to Theorem 7.1.1, this integral equals

$$\begin{aligned}
&C\int_\rho^R \left(\int_r^R C_j^{1/2}(s/r)F_{j,m}^{(2)}(s)ds\right)^2 dr \\
&\le C(1+2R/\rho)^{2j}\int_\rho^R \left(\int_r^R |F_{j,m}^{(2)}(s)|ds\right)^2 dr \\
&\le C(1+2R/\rho)^{2j}\int_\rho^R (R-r)\int_r^R (F_{j,m}^{(2)}(s))^2 ds)dr \\
&\le C(1+2R/\rho)^{2j}(R-\rho)^2\int_\rho^R (F_{j,m}^{(2)}(s))^2 ds.
\end{aligned}$$

In the second inequality we made use of (7.1.10), and then of Fubini's theorem and Hölder's inequality. Using the interpolation theorem for spaces $H_{(k)}(\rho, R)$ and then the definition of the norm $\|\ \ \|_{\bullet,2}$ and orthonormality of spherical harmonics on the unit sphere, we obtain

$$\begin{aligned}
\int_\rho^R (F_{j,m}^{(2)}(s))^2 ds &\le C\left(\int_\rho^R (F_{j,m}^{(3)}(s))^2 ds\right)^{1/3}\left(\int_\rho^R (F_{j,m}(s))^2 ds\right)^{2/3} \\
&\le C\left(\int_{|\sigma|=1}\int_{\mathbb{R}} (F_{j,m}^{(3)}(s)Y_{j,m}(\sigma))^2\, ds\, d\sigma\right)^{1/3} \|F\|_2^{4/3}(A) \\
&\le C\|F\|_{\bullet,3}^{2/3}\|F\|_2^{4/3}(A) \\
&\le C\|f\|_{(2)}^{2/3}\|F\|_2^{4/3}(A) \le CM^{2/3}\|F\|_2^{4/3}(A)
\end{aligned}$$

by Theorem 7.1.4. We conclude that

$$\int_\rho^R f_{j,m}^2 r^{n-1} dr \le Ce^{2Lj}M^{2-\lambda}\|F\|_2^\lambda(A), \quad L = \ln(1+2R/\rho) \tag{7.1.11}$$

when $n = 3$ with $\lambda = 4/3$.

This integral in the two-dimensional case is

$$\int_\rho^R \left(\int_r^R (s^2-r^2)^{-1/2} C_j^0(s/r) F_{j,m}^{(1)}(s)\,ds\right)^2 dr$$
$$\le Ce^{2Lj}\left(\int_\rho^R \left(\int_r^R (s^2-r^2)^{-3/4}\,ds\right)^{4/3} dr\right)\left(\int_\rho^R (F_{j,m}^{(1)}(s))^3\,ds\right)^{2/3}$$
$$\le Ce^{2Lj}\left(\int_\rho^R (F_{j,m}^{(1)}(s))^3\,ds\right)^{2/3},$$

where as above we have used (7.1.10) and the Hölder inequality for the integral with respect to $s$. The last integral by the embedding theorem (with $q=3, k=1, m=\frac{7}{6}, n=1, p=2$) is less than

$$C\|F_{j,m}\|^2_{(7/6)}(\rho,R) \le C\|F_{j,m}\|^{14/9}_{3/2}\|F_{j,m}\|^{4/9}_{(0)}(\rho,R) \le C\|F\|^{14/9}_{\bullet,3/2}\|F\|^{2/3}_{(0)}(A)$$
$$\le C\|f\|^{14/9}_{(1)}\|F\|^{4/9}_{(0)}(A) \le CM^{14/9}\|F\|^{4/9}_2(A),$$

where in the second inequality we have used the interpolation theorem with $\theta = 2/9, s_1 = 3/2, s_2 = 0$. Thus, we obtain the bound (7.1.11) when $n=2$.

Now we can complete the proof as in Section 6.1. Observe that

$$\sum_{j,m}\int_0^R j^2 f_{j,m}^2 r^{n-1} dr \le C\|f\|^2_{(1)}(B) \le CM^2.$$

By orthonormality and completeness of spherical harmonics,

$$\|f\|_2^2(A) = \sum_{m,j<J}\int_\rho^R f_{j,m}^2 r^{n-1}dr + \sum_{m,J\le j}\int_\rho^R f_{j,m}^2 r^{n-1}dr$$
$$\le Ce^{2LJ}M^{2-\lambda}\|F\|_2^\lambda(\mathcal{A}) + CJ^{-2}M^2,$$

where we have made use of (7.1.11) and of the previous observation. Let us choose $J$ from the inequalities $2J \le -\delta \ln \|F\|_2(\mathcal{A}) \le 2J+2$, where positive $\delta$ is to be determined later. Then the previous inequality yields

$$\|f\|_2^2(A) \le CM^2(\|F\|_2^{-\delta L+\lambda}(\mathcal{A}) + 1/(-\delta/2 \ln \|F\|_2(\mathcal{A}) - 1)).$$

Choosing $\delta < \lambda/L$ and using that the logarithm grows more slowly than any positive power, we complete the proof. □

It is realistic that one can derive stability estimates from a singular value decomposition for the Radon transform given in the most advanced form by Quinto [Q].

Lissianoi [Lis] showed the one cannot improve logarithmic estimates of Theorem 7.1.4 to Hölder even when looking for $f$ in $\{R+\varepsilon < |x|\}$ while $F$ is given on $\{R < |s|\}$. This is contrast to interior estimates for the Cauchy problem in differential equations and shows that it is not possible to apply to integral geometry over straight lines and planes reduction to hyperbolic equations and use of Carleman-type estimates. The spherical means tranform defined as integrals of $f$

over a family of spheres centered at $\gamma$ with radius changing from 0 to $T$ has better stability properties: indeed it is closely related to the Cauchy problem for the wave equation with the data on the lateral surface $\gamma \times (0, T)$. This Cauchy problem is discussed in sections 3.2, 3.4. There is a renewed interest to the spherical transform due to applications in ultrasound detection [FPR].

Despite the sharp estimates of Theorem 7.1.4, the Radon transform is not an invertible map from $H_{(\alpha)}$ onto $H_{\bullet,\alpha+(n-1)/2}$ because its range is substantially smaller than this space. There is a complete description of the range given by the following Helgason-Ludwig consistency conditions:

$$\int_{\mathbb{R}} s^m F(\sigma, s) ds \text{ is a homogeneous polynomial of degree } m \text{ in } \sigma,$$

$$F(\sigma, s) = F(-\sigma, -s). \tag{7.1.12}$$

A similar characterization was given by Kuchment and Lvin in 1990 for the attenuated Radon transform in the plane when the attenuation coefficient is constant, and we refer for some generalizations to the recent paper of Aguilar, Ehrenpreis, and Kuchment [AEK]. It claims that $F_\mu$ is the exponential Radon transform of a function $f \in C_0^\infty(\mathbb{R}^2)$ if and only if $F_\mu \in C_0^\infty(\mathbb{R} \times S^1)$ and

$$\hat{F_\mu}(\sigma(s), is) = \hat{F_\mu}(\sigma(-s), -is),$$

where $\sigma(s)$ is $\sigma$ rotated clockwise by the angle $\arcsin s(s^2 + \mu^2)^{1/2}$.

We define the weighted Radon transform of $f$ (with weight $\rho$) as

$$R_\rho f(\omega; \omega \cdot x) = \int_{(y-x)\cdot\omega=0} \rho(y, x; \omega) f(y) d\Gamma(y) \tag{7.1.13}$$

and observe that the constant attenuation in $\mathbb{R}^2$ with the attenuation coefficient $\mu > 0$ means that

$$\rho(y, x, \omega) = \exp(\mu(y - x) \cdot (-\omega_2, \omega_1)).$$

Another inversion tool is the averaging operator $S$ with respect to $\omega$:

$$SF = \int_{|\omega|=1} F(\omega) d\omega.$$

**Lemma 7.1.6.**

$$SR_\rho f(x) = \int |x - y|^{-1} \rho^\bullet(y, x) f(y) dy, \tag{7.1.14}$$

*where*

$$\rho^\bullet(y, x) = \int_{|\omega|=1, (x-y)\cdot\omega=0} \rho(y, x; \omega) d\omega.$$

PROOF. By definition,

$$SR_\rho f(x) = \int_{|\omega|=1} \int_{(y-x)\cdot\omega=0} \rho(y, x; \omega) f(y) d\Gamma(y) dS(\omega)$$

By introducing the polar coordinates $y = x + r\nu$, $\nu \cdot \omega = 0$ in the hyperplane $(y - x) \cdot \omega = 0$, we write the interior integral as

$$\int_{|\nu|=1, \nu\cdot\omega=0} \int_0^{+\infty} r^{n-2} \rho(x + r\nu, x; \omega) f(x + r\nu) dr\, dS(\nu).$$

By interchanging the order of integration with respect to $\omega$ and $\nu$ we obtain

$$\begin{aligned} &SR_\rho f(x) \\ &= \int_{|\nu|=1} \int_0^{+\infty} r^{-1} \\ &\times \left( \left( \int_{|\omega|=1, \omega\cdot\nu=0} \rho(x + r\nu, x; \omega) \right) dS(\omega) f(x + r\nu) r^{n-1} dr dS(\nu) \right). \end{aligned}$$

Returning from polar coordinates to $y = x + r\nu$ and using that $dy = r^{n-1} dr d\nu$, $r = |x - y|$, and $\omega \cdot \nu = 0$ is equivalent to $(x - y) \cdot \omega = 0$, we obtain the formula (7.1.14). □

When $\rho = 1$ (the classical Radon transform) we have

$$SR_1 f(x) = c_n \int |x - y|^{-1} f(y) dy.$$

In particular, if $n = 3$, then the right side is the volume potential of $f$; so applying the Laplace operator to the both parts, we will find that

$$f(x) = c\Delta_x \int_{|\omega|=1} R_1(x, \omega) d\omega, \tag{7.1.15}$$

where $c$ can be shown to be $-1/(8\pi^2)$. This formula was known already to Radon. In a more general situation we can reduce the problem of reconstruction of $f$ to a Fredholm integral equation and obtain Schauder-type estimates for $f$.

**Corollary 7.1.7.** *Let $n = 3$ and $\rho \in C^3(\overline{\Omega} \times \overline{\Omega} \times \Sigma)$. Then the inversion of the attenuated Radon transform can be obtained as a solution of the following Fredholm integral equation in $C(\overline{\Omega})$:*

$$cf(x) + \int_\Omega K(x, y) f(y) dy = (-\Delta) SF(x), \tag{7.1.16}$$

*where*

$$K(x, y) = 2\nabla_x |x - y|^{-1} \cdot \nabla_x \rho^\bullet(x, y) + |x - y|^{-1} \Delta_x \rho^\bullet(x, y),$$

*We have the Schauder-type estimate*

$$|f|_\lambda(\Omega) \le C(|SF|_{2+\lambda}(\Omega) + |f|_0(\Omega)).$$

PROOF. By using that $-\Delta_x |x - y|^{-1} = 4\pi\delta(x)$, we get from (7.1.14)

$$cf(x) + \int_\Omega K(x, y) f(y) dy = (-\Delta) SF(x).$$

Due to our assumptions on $\rho$, this kernel satisfies the conditions

$$|K| \leq C|x-y|^{-2}, \quad |\nabla_x K| \leq C|x-y|^{-3},$$

so it is an $N^{(1,1)}$-kernel by [Mi], p. 23. As shown in the book of Miranda ([Mi], p. 27), the integral operator with kernel $K$ is continuous from $C(\overline{\Omega})$ into $C^{\lambda}(\overline{\Omega})$ and for any $\lambda$, $0 < \lambda < 1$. Since the embedding of $C^{\lambda}$ into $C$ is compact, the equation is Fredholm. The estimate of Corollary 7.1.7 also follows from the above-mentioned continuity of this operator. □

We observe that the weighted Radon transform is defined for a more general family of hypersurfaces than hyperplanes. In the papers of Bukhgeim and Lavrentiev [LaB] and of Beylkin [Bey] there are results similar to Corollary 7.1.7 in a more general situation and also for the plane case, where on has to use $(-\Delta)^{1/2}$, which is a first-order pseudo-differential operator.

When vol $\Omega$ is small, the same argument implies that the integral operator is a contradiction, and therefore we have uniqueness and stability of inversion of the attenuated Radon transform.

As a useful fact we give (without proof, which one can find in the paper of Isakov and Sun [IsSu1]) the following uniqueness and stability results on the weighted X-ray transform

$$P^{\bullet}_{\rho} f(x,\sigma) = \int_{l(x,\sigma)} \rho(l,\,) f \, dl,$$

where $l(x,\sigma)$ is the straight line passing through $x$ with direction $\sigma$. To formulate it, we introduce a ball $B$ in $\mathbb{R}^3$ and the manifold $\Pi(h) = \{(x,\sigma)$ : the intersection of the plane through $x$ with the normal $\sigma$ with $B$ is at positive distance from $\{x_3 \geq -h\}\}$. Let $B^-(h)$ be $B \cap \{x_3 < -h\}$.

**Theorem 7.1.8.** *Let the weight function $\rho \in C^2(\mathbb{R}^6)$ and $\rho > \delta > 0$. Then there is a constant $C(\delta, |\rho|_2)$ such that*

$$|f|_0(B^-(h+\varepsilon) \leq C\varepsilon^{-4}|P^{\bullet}_{\rho} f|_{3/2}(\Pi(d))$$

*for all functions $f \in C^2_0(B)$.*

## 7.2 The energy integral methods

Let $\gamma(x,y)$ be a $C^2$-smooth regular curve with endpoints $x, y \in \overline{\Omega} \subset \mathbb{R}^2$. We assume that the family of curves $\{\gamma(x,y)\}$ has the following properties: (a) For any $x, y \in \overline{\Omega}$ there is a unique $\gamma(x,y)$. (b) This family is locally diffeomorphic to the family of straight lines between the same points. We refer for more detail concerning condition (b) to the papers of Muhometov [Mu1], [Mu2] and the book of Romanov [Rom]. Let $y(x_3)$ be a parametrization of $\partial\Omega$, $x_3 \in I = [0, S]$. Let us introduce the function

$$F(x, y) = \int_{\gamma(x,y)} \rho(v, y) f(v) d\gamma(v). \tag{7.2.1}$$

Let $\rho^*(v, x_3) = \rho(v, y(x_3))$ and $F^*(x, x_3) = F(x, y(x_3))$. Let $\tau = (\cos\phi, \sin\phi)$ be the tangent direction to $\gamma(x, y(x_3))$ at a point $x$.

The following result is due to Muhometov [Mu1].

**Theorem 7.2.1.** *Assume that the weight function $\rho$ satisfies the following conditions: $\rho^* \in C^2(\overline{\Omega} \times I)$, $\rho^* > \varepsilon > 0$, and*

$$|\partial_3 \ln \rho^*| < \partial_3\phi \textit{ on } \overline{\Omega} \times I. \tag{7.2.2}$$

*Then*

$$\|f\|_2(\Omega) \le C\|\nabla F^*\|_2(\partial\Omega \times I), \tag{7.2.3}$$

*where $C$ depends on the family $\{\gamma\}$ and on $\rho$. Moreover, $C = (2\pi)^{-1/2}$ when $\rho = 1$.*

PROOF. Let us consider the function

$$u(x_1, x_2, x_3) = \int_{\gamma((x_1,x_2), y(x_3))} \rho(y, x_3) f(y) d\gamma(y). \tag{7.2.4}$$

This function satisfies the first-order partial differential equation

$$\tau_1\partial_1 u + \tau_2\partial_2 u = \rho f \quad \text{in } \Omega \times I. \tag{7.2.5}$$

The function $f$ does not depend on $x_3$, so dividing both parts by $\rho$ and differentiating with respect to $x_3$ we get the homogeneous second-order equation

$$\partial_3(\rho^{-1}\tau_1\partial_1 u + \rho^{-1}\tau_2\partial_2 u) = 0 \text{ in } \Omega \times I. \tag{7.2.6}$$

To make use of the energy integrals method we multiply this equation by the nonstandard factor $(-\rho\tau_2\partial_1 u + \rho\tau_1\partial_2 u)$ and integrate over $\Omega \times I$, observing that $\rho\partial_3\rho^{-1} = -\partial_3 \ln \rho$. We obtain

$$\begin{aligned} 0 = \int_{\Omega\times I} &((\partial_3 \ln \rho)(\tau_2\partial_1 u - \tau_1\partial_2 u)(\tau_1\partial_1 u + \tau_2\partial_2 u) \\ &+ (-\tau_2\partial_1 u + \tau_1\partial_2 u)(\partial_3\tau_1\partial_1 u + \partial_3\tau_2\partial_2 u) \\ &+ (-\tau_2\partial_1 u + \tau_1\partial_2 u)(\tau_1\partial_3\partial_1 u + \tau_2\partial_3\partial_2 u)). \end{aligned}$$

Using the notation $\tau^\perp = (\tau_2, -\tau_1)$, multiplying the terms in parentheses and integrating by parts the term

$$-\tau_2\tau_1\partial_1 u\partial_3\partial_1 u = -\tau_1\tau_2\partial_3(\partial_1 u)^2,\ \tau_1\tau_2\partial_2 u\partial_3\partial_2 u = \tau_1\tau_2\partial_3(\partial_2 u)^2/2$$

of the last parentheses, and using that due to the $S$-periodicity with respect to $x_3$ there will be no boundary terms resulting from this integration by parts, we obtain

$$\int_{\Omega\times I}(\partial_3 \ln\rho)\tau^{\perp}\cdot\nabla u\tau\cdot\nabla u$$

$$= \int_{\Omega\times I}(-\tau_2\partial_3\tau_1(\partial_1 u)^2 + (\tau_1\partial_3\tau_1 - \tau_2\partial_3\tau_2)\partial_1 u\partial_2 u + \tau_1\partial_3\tau_2(\partial_2 u)^2$$

$$+\frac{1}{2}\partial_3(\tau_1\tau_2)(\partial_1 u)^2 + \tau_1^2\partial_2 u\partial_1\partial_3 u - \tau_2^2\partial_1 u\partial_2\partial_3 u - \frac{1}{2}\partial_3(\tau_1\tau_2)(\partial_3 u)^2) \tag{7.2.7}$$

Integrating by parts with respect to $x_3$ in the term $\tau_1^2\partial_2 u\partial_1\partial_3 u$, we obtain the term $-2\tau_1\partial_3\tau_1\partial_2 u\partial_1 u - \tau_1^2\partial_2\partial_3 u\partial_1 u$. Collecting all terms involving $\partial_1 u\partial_2 u$, we obtain the term $-(\tau_1\partial_3\tau_1 + \tau_2\partial_3\tau_2)\partial_1 u\,\partial_2 u$, which is zero because the factor in the parentheses is $\frac{1}{2}\partial_3(\tau_1^2+\tau_2^2) = \frac{1}{2}\partial_3 1 = 0$. So the last integral is reduced to

$$\int_{\Omega\times I}(\frac{1}{2}(\tau_1\partial_3\tau_2 - \tau_2\partial_3\tau_1)((\partial_1 u)^2 + (\partial_2 u)^2) - (\tau_1^2+\tau_2^2)\partial_1 u\partial_2\partial_3 u).$$

Now we will make use of the identity $\tau_1^2+\tau_2^2 = 1$, the identities $\partial_3\tau_1 = -\tau_2\partial_3\phi$, $\partial_3\tau_2 = \tau_1\partial_3\phi$, and the relations

$$-\int_{\Omega\times I}\partial_1 u\partial_2\partial_3 u = -\frac{1}{2}\int_{\Omega\times I}\partial_1 u\partial_2\partial_3 u - \frac{1}{2}\int_{\partial\Omega\times I}\partial_1 u\partial_3 u\nu_2$$
$$+\frac{1}{2}\int_{\partial\Omega\times I}\partial_2 u\partial_3 u\nu_1 - \frac{1}{2}\int_{\Omega\times I}\partial_2 u\partial_1\partial_1\partial_3 u.$$

The sum of the first and the last integrals on the right side is zero because $\partial_1 u\partial_2\partial_3 u + \partial_2 u\partial_1\partial_3 u = \partial_3(\partial_1 u\partial_2 u)$, so the integral of this function over $\Omega\times I$ is zero due to the $S$-periodicity with respect to $x_3$.

Summing up and using that $(\tau_1\partial_3\tau - \tau_2\partial_3\tau_1) = (\tau_1^2+\tau_2^2)\partial_3\phi$, from (7.2.7) we conclude that

$$\int_{\Omega\times I}\partial_3\phi((\partial_1 u)^2 + (\partial_2 u)^2) + 2\int_{\Omega\times I}(\partial_3\ln\rho)\tau^{\perp}\cdot\nabla u\tau\cdot\nabla u$$
$$= -\int_{\partial\Omega\times I}(-\nu_2\partial_1 u + \nu_1\partial_2 u)\partial_3 u.$$

Due to the assumption (7.2.2), $\partial_3\phi > |\partial_3\ln\rho| + \varepsilon$ on $\Omega\times I$. By using this inequality as well as the known inequality

$$-2|\tau^{\perp}\cdot\nabla u\tau\cdot\nabla u| \geq -|\tau^{\perp}\cdot\nabla u|^2 - |\tau\cdot\nabla u|^2$$
$$= -((\partial_1 u)^2 + (\partial_2 u)^2),$$

we conclude that

$$\varepsilon\int_{\Omega\times I}((\partial_1 u)^2 + (\partial_2 u)^2) \leq \int_{\partial\Omega\times I}|\nu^{\perp}\cdot\nabla u||\partial_3 u|.$$

From equation (7.2.5) we have $|f|^2 \leq |\rho|^{-2}((\partial_1 u)^2 + (\partial_2 u)^2)$. In addition, $\nu^{\perp}\cdot\nabla u$ is the tangential component of $\nabla u$, so the last integral is bounded by $C\|u\|_{(1)}(\partial\Omega\times I)$.

Thus we obtain the bound (7.2.3) in the general case.

When $\partial_3\rho = 0$ we can repeat the previous argument and use the relations

$$\int_{\Omega\times I} \partial_3\phi((\partial_1 u)^2 + (\partial_2 u)^2) \geq \int_{\Omega\times I} \partial_3\phi\rho^2 f^2$$
$$= \int_\Omega \rho^2 f^2 \left(\int_I \partial_3\phi dx_3\right) dx = 2\pi \int_\Omega \rho^2 f^2,$$

which follow from equation (7.2.5) and the independence of $f$ on $x_3$, to complete the proof □

This proof essentially belongs to Muhometov. In fact, Theorem 7.2.1 is only formulated in his paper [Mu1], where the basic idea of the given proof is actually applied to a more difficult problem of determining a Riemannian metric from the lengths of its geodesics, as described below. More detail are given in [Mu2]. One of the crucial steps is to find an appropriate factor by which to multiply equation (7.2.5). It seems that this multiplying factor has origin in Friedrich's theory of symmetric positive systems [Fri] and that it is unique. Multidimensional versions are given by Beylkin, Gerver, Markushevich, Muhometov, and Romanov [Rom]. They are quite similar to Theorem 7.2.1, and their main condition is that $\gamma(x, y)$ are geodesics of a Riemannian $C^2$-metric that form a regular family in $\overline{\Omega}$ and $\Omega$ is convex with respect to these geodesics. Regularity of geodesics loosely speaking means that locally their family is diffeomorphic to the family of straight lines.

For other possibilities of the two-dimensional case we refer to the paper of Gelfand, Gindikin, and Shapiro [GeGS].

Vector (and tensor) versions of this theory are obtained and presented in the book of Sharafutdinov [Sh] where one can find a necessary preliminary on Riemannian geometry and multidimensional versions of Mukhometov's theory, as well as many applications to problems of geophysics and vector tomography. We will briefly describe some of recent findings of Pestov and Uhlmann [PU] based on this theory.

A compact Riemannian manifold $\Omega$, $g$ with the boundary is simple if $\Omega$ is simply connected, any geodesic has no conjugate points, and $\partial\Omega$ is strictly convex (in the metric $g$). Then the distance $d(x, y)$ between two points $x, y \in \overline{\Omega}$ is well defined. Let $d_j$ corresponds to the metric $g_j$, $j = 1, 2$.

**Theorem 7.2.2** ([PU]). *Let* $(\Omega, g_j)$, $j = 1, 2$ *be two two-dimensional simple Riemannian manifolds with the boundary. Let*

$$d_1(x, y) = d_2(x, y), \text{ when } (x, y) \in \partial\Omega \times \partial\Omega$$

*Then there is a diffeomorphism* $\Phi$ *of* $\overline{\Omega}$ *onto itself identical on* $\partial\Omega$ *and such that* $g_2 = \Phi^* g_1$.

We remind that $g_2 = \Phi^* g_1$ if $g_2(x)\xi \cdot \eta = g_1(\Phi(x))(\Phi'(x)\xi) \cdot (\Phi'(x)\zeta)$ for all vectors $\xi, \eta$ of tangent space to $\Omega$ at all $x \in \Omega$.

We mention ideas of the proof in [PU]. Sharafutdinov [Sh] showed that the conclusion of Theorem 7.2.2 follows from Theorem 7.2.2 with the additional assumption that $g_1 = g_2$ on $\partial\Omega$. By using the so-called scattering relation they showed in [PU] that the conditions of the weaker form of Theorem 7.2.2 imply equality of the Dirichlet- Neumann maps of the Laplace-Beltrami equations corresponding to $g_1$ and $g_2$. Now Theorem 5.5.7 of Lassas and Uhlmann and of Belishev combined with known facts from two-dimensional Riemannian geometry imply existence of the needed diffeomorphism.

Observe that in the three-dimensional case a uniqueness result for the attenuated X-ray transform can be obtained only under positivity and minimal smoothness assumptions with respect to $\rho$. This is clear from Theorem 7.1.8 because in this case we can slice a general domain $\Omega$ by plane domains of small volume and use uniqueness in the plane case for domains of small diameter.

The interesting counterexample given in the next section shows that without condition (7.2.2) Theorem 7.2.1 is wrong.

## 7.3 Boman's counterexample

To describe the following example of nonuniqueness we will make use of a slightly different parametrization of a weight function $\rho(y, L)$ and of a line $L$ itself. $L = \{y : y \cdot \omega = p\}$ is identified with the unit vector $\omega$ and the real number $p$, so the manifold of the straight lines is considered as $\Sigma \times \mathbb{R}$. By using an appropriate parametrization of $L$ one can obtain a function $\rho$ of variables used in Theorem 7.2.1.

Boman [Bo] found the following convincing counterexample.

**Theorem 7.3.1.** *Let $B$ be the unit disk in $\mathbb{R}^2$. There are functions $\rho(x, L)$ and $f(x)$ such that $\rho \in C^\infty$, $\rho > \rho_0 > 0$, $f \in C^\infty(\overline{B})$, $f \neq 0$, and*

(7.3.1)
$$\int_{L\cap B} \rho(y, L) f(y) dL(y) = 0$$

*for any straight line $L$.*

Proof. We first construct

(7.3.2)
$$f = \sum f_k/k!,$$

where

(7.3.3)
$$f_k(r, \theta) = \phi(2^k(1 - r)) \cos(3^k\theta), k = 1, 2 \ldots,$$

and $\phi$ is a $C^\infty$-function such that $0 \leq \phi \leq 1$. It is zero outside the interval $(\frac{4}{5}, \frac{6}{5})$, and it is positive on this interval. Here $r, \theta$ are the polar coordinates in the plane. because of the factor $k!$ in the denominators, the series (7.3.2) and all its partial derivatives are uniformly convergent on $B$, which implies that $f$ is $C^\infty$-smooth. Observe that $f = 0$ when $r < \frac{2}{5}$, and that supports of all $f_k(r, \theta)$ are disjoint annuli accumulating to $\partial B$.

*The weight function near $\partial B$.*

To construct the weight function $\rho_0$ we make use of a partition of unity $\{\psi_k\}$ of the interval $\frac{1}{2} < r < 1$ such that $\psi_k$ is zero outside $(1-2^{-k+1}, 1-2^{-k-1})$ and $\|\partial^m \psi_k\|_\infty \leq C(m)2^{km}$. To find $\psi_k$ one can use translations, scaling, and well-known results on cutoff functions (e.g., see [Hö2], Chapter 1). We define

$$\rho_0(x, L) = 1 - \sum_{k=3}^{\infty} A_k(L) f_k(x), \tag{7.3.4}$$

where

$$A_k(L) = k! \quad \psi_{k-2}(|p|) \int_L f \Big/ \int_L f_k^2, \tag{7.3.5}$$

and we have used the standard parametrization of the straight line $L$ as $\{y : y \cdot \omega = p\}$.

**Exercise 7.3.2.** Prove that the function $\rho = \rho_0$ defined by formula (7.3.4) satisfies condition (7.3.1)

*Smoothness of the weight function.*

Since for any $p$ in $(-1, 1)$ the series (7.3.4) contains at most two nonzero terms, it is obvious that $\rho$ is smooth for such $p$. So to establish smoothness for $|p| \leq 1$ it suffices to show that the sum in (7.3.4) and all its derivatives with respect to $x_j$, $p$, and $\omega$ converge to zero when $|p|$ goes to 1. We write the sum as

$$B(L)\Sigma_{k\geq 3} k! \quad f_k(x)\psi_k(|p|)/C_k(L),$$

where

$$B(L) = \int_L f, \qquad C_k(L) = \int_L f_k^2.$$

We claim that the following bounds are valid:

$$|\partial^\alpha f_k| \leq C(\alpha)3^{|\alpha|k}, \tag{7.3.6}$$

for any $m$ there are $C(\beta, m)$ such that

$$|\partial^\beta B| \leq C(\beta, m)2^{-mk}/k!, \text{ provided that } 1 - |p| < 2^{-k}; \tag{7.3.7}$$

and

$$|\partial^\beta(\psi_k(|p|)/C_k(L))| \leq C(\beta)3^{2|\beta|k}, \tag{7.3.8}$$

and $\partial^\alpha$ mean differentiation with respect to $x$, while $\partial^\beta$ is differentiation with respect to $\omega$, $p$.

The bound (7.3.6) is obvious.

To obtain (7.3.7) we will make use of the coordinates $p$ and $\sigma$, $\omega = (\cos\sigma, \sin\sigma)$, of a line $L$. We have $B = \sum B_k/k!$, where $B_k$ is the integral of $f_k$ over $L$. We write $B_k$ by using the parametrization of $L$ by the angle $u = \theta - \sigma$, where $\theta$ is the angular polar coordinate in the plane. We have

$$B_k = \int_{-\pi/2}^{\pi/2} \phi(2^k(1 - p/\cos u)) \cos 3^k(\sigma + u) \cos^{-2} u|p|du$$
$$= |p|3^{-kN} \int_{-\pi/2}^{\pi/2} \partial_u^N (\phi(2^k(1 - p/\cos u))/\cos^2 u) \cos(3^k(\sigma + u))du$$

when we integrate by parts. This gives the bound $|B_k| \le C(N)(2/3)^{Nk}$. Using that $B_k$ is zero when $1 - |p| < 2^{-k-1}$, summing over $k^\# > k$, and choosing $N$ such that $(\frac{2}{3})^N < 2^{-m}$, we obtain the bound (7.3.7) when $\beta = 0$. To prove the general case we differentiate and repeat the argument.

The bound (7.3.8) can be obtained similarly. The bound $|\partial^\beta C_k| \le C(\beta)3^{2|\beta|k}$ can be obtained exactly as for $B_k$. In fact, it is easier because we do not need to integrate by parts. Observe that $|C_k| \ge 2^{-k}/C$ because when $1 - |p| \ge 2^{-k+1}$, the line $L$ intersects annular supports of $f_k^2$ over intervals of length $2^{-k}/C$. Now the bound (7.3.8) follows by applying the differentiation formulae for products and quotients.

Since the supports of the functions $f_k$ do not intersect, at any particular point of $B$ there is only one zero term of the series (7.3.4) as well as of the differentiated series, so $C^\infty$-smoothness when $|p| < 1$ and $x \in B$ is obvious. From the bounds (7.3.6)–(7.3.8) it follows that any such derivative $\partial^\alpha \partial^\beta$ of the sum of this series goes to 0 when $|p|$ is approaching 1. Indeed, when $1 - |p| < 2^{-k}$ we have only two nonzero functions $\psi_k$, $\psi_{k+1}$, and accordingly only the terms of the sum (7.3.4) with the indices $k + 2, k + 3$. From the formulae for differentiation of products and the bounds (7.3.6)–(7.3.8) we conclude that the $\partial^\alpha \partial^\beta$ of this sum is bounded by the sum of

$$C(\alpha, \beta, m)3^{2(|\alpha|+|\beta|)k}2^{-mk} < C(\alpha, \beta, m)2^{-k}$$

when we choose $m$ such that $3^{2(|\alpha|+|\beta|)} < 2^{m-1}$. Now the convergence to 0 is obvious.

*Positivity of the weight function.*

From the previous results it follows that $\rho_0 \to 1$ when $|p| \to 1$, so

$$\rho_0 > \frac{1}{2} \text{ when } |p| > 1 - p_0. \qquad (7.3.9)$$

At this stage we have built $\rho$ for lines $L$ close to $\partial B$. Now we complete the construction for all lines $L$, which is much easier to do for $L$ close to the origin.

*Lines close to the center of $B$.*

The definition (7.3.2), (7.3.3) guarantees that the function $f$ changes sign on any straight line $L$ inside $B$.

Indeed, if $L$ does not cross 0, then for large $k$, supp $f_k$ intersects $L$ over two intervals. The angular size of each interval is greater than $2^{-k}/C(L)$. On the other hand, the period of the angular component of $f_k$ is $2\pi 3^{-k}$, so for large $k$ the angular size of the interval is greater than the period of the angular part, so this part changes sign on this interval. If $L$ crosses 0, then its points in polar coordinates are $(r, \gamma)$, 0, and $(r, \gamma + \pi)$, where $\gamma$ is fixed and $r$ varies between 0 and 1. Then our claim follows from the relation $\cos(3^k(\gamma + \pi)) = -\cos(3^k\gamma)$, so $f(r, \gamma + \pi) = -f(r, \gamma)$.

Hence for any $L_0$ there is a nonnegative function $\chi \in C^\infty$ such that $\int_{L_0} f\chi dL_0 \neq 0$, and if $\int_{L_0} f dL_0 \neq 0$, then the first integral has the opposite sign. The function

$$\rho(x, L) = 1 - \chi(x) \int_L f dL / \int_L f\chi dL$$

will be $C^\infty$ with respect to all variables ($x$ and $L$) and $\geq \frac{1}{2}$ when $L$ is in a neighborhood $V(L_0)$ of $L_0$.

*Final construction of $\rho$.*

To complete the construction in the whole of $B$ we use a partition of unity in the manifold of lines. We already have the function $\rho_0 > \frac{1}{2}$ when $|p| > p_0$. Let $\mathcal{L}$ be the set of all straight lines $L$ with $|p| \leq p_0$. This is a compact set covered by open sets $V(L)$. Choose a finite subcovering $V_1, \ldots, V_N$ of $\mathcal{L}$, and denote the weight functions around lines in $V_j$ by $\rho_j(x, L)$. Define $V_0$ as $\{p_0 \leq |p| \leq 1\}$. Let $\phi_p, \ldots, \phi_N$ be a nonnegative $C^\infty$-partition of unity subordinated to the covering $V_0, \ldots, V_N$ of the set of lines $\{|p| \leq 1\}$. In particular, $\operatorname{supp} \phi_j \subset V_j$. Finally, we define

$$\rho(x, L) = \sum \phi_j(L)\rho_j(x, L).$$

Obviously, $\rho > \frac{1}{2}$, and it is in $C^\infty$. Since all functions $\rho_j$ satisfy the relation (7.3.1), it is easy to see that so is $\rho$.

The proof is complete. □

This proof is a slight modification of the one in [Bo].

## 7.4 The transport equation

Of significant applied interest is the following integro-differential equation

$$\partial_t u + v \cdot \nabla u + b_0 u = \int_W K(x, v, w, t)u(x, w, t)dw + f(x, v, t),$$
$$u = u(x, t, v) \tag{7.4.1}$$

augmented by the initial conditions

$$u = u_0 \text{ on } \Omega \times \{0\} \times W \tag{7.4.2}$$

and the boundary conditions

$$u = g_- \text{ on } \Gamma_- \times (0, T), \tag{7.4.3}$$

where $\Gamma_-$ is the part $\cup(\partial\Omega \cap \{\nu \cdot v < 0\} \times \{v\})$ of $\partial\Omega \times W$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega \in C^1$, and $W \subset \mathbb{R}^n$ is the velocity domain. Quite often $W = S^{n-1}$.

Assuming that $\partial\Omega \in C^2$; $u_0 \in L_2(\Omega \times W)$; $g_- \in C^1([0, T]; L_2(\Gamma_-)$; $b_0 \in L_\infty(\Omega \times W)$; $K$ is a bounded, measurable, nonnegative function of all its variables satisfying the condition

$$\int_{W\times W} |K(x, v, w)|^2 dv\, dw \le k;$$

and $f \in C([0, T]; L_2(\Omega \times W))$, one can prove uniqueness and existence of a solution $u \in C^1([0, T], L_2(\Omega \times W))$ of the initial boundary value problem (7.4.1)–(7.4.3).

The natural inverse problem for equation (7.4.1) is to determine the coefficient $b_0$, the kernel $K$ of the collision operator, and the density of interior radiation sources $f$ from the lateral data

(7.4.4) $$u = g_+ \text{ on } \Gamma_+ = (\partial\Omega \times W \backslash \Gamma_-) \times (0, T)$$

and final observations

(7.4.5) $$u = u_T \text{ on } \Omega \times \{T\} \times W.$$

We will describe some of them below.

All possible lateral boundary measurements can be viewed as the so called albedo operator $\Lambda$ which is similar to the Dirichlet-to Neumann operator for partial differential operators of second order. We define $\Lambda(g_-) = g_+$ given by (7.4.4) where $u$ solves the initial boundary value problem (7.4.1) – (7.4.3) with $f = 0, u_0 = 0$. From the mentioned results on the direct problem (7.4.1) – (7.4.3) it follows that $\Lambda$ is a linear continuous operator from $C^1([0, T]; L_2(\Gamma_-)$ into $C^1([0, T]; L_2(\Gamma_+)$. It is known that $\Lambda$ has the continous extension as the operator from $L_1((0, T); L_{1;\nu\cdot v}(\Gamma_-)$ into $L_1((0, T); L_{1;\nu\cdot v}(\Gamma_+)$ where $L_{1;\nu\cdot v}(\Gamma)$ is the weighted $L_1(\Gamma)$-space with the weight $|\nu(x) \cdot v|$.

It was shown by Choulli and Stefanov [ChoS1] that if $\Omega$ is convex then the albedo operator with $T = +\infty$ has the Schwarz kernel $A(t - \theta, x, v, y, w)$ which is the sum of the two terms

$$|\nu(y) \cdot w|^{-1} exp\Big( - \int_0^{\tau_(x,v)} b_0(x - sv, v)ds)\delta(y - (x - \tau_-(x, v)v); \Gamma_-)$$
$$\times\, \delta(v - w)\delta(\tau - \tau_-(x, v)\Big),$$

$$|\nu(y) \cdot w|^{-1} \int exp\Big( - \int_0^s b_0(x - pv, v)dp$$
$$- \int_0^{\tau_-(x-sv,w)} b_0(x - sv - pw, w)dp)\delta(\tau - s - \tau_-(x - sv, w)\Big)$$

$$K(x - sv, v, w)\delta(y - (x - sv - \tau_-(x - sv, w)w); \Gamma_-)\chi(x - sv, x)ds$$

and of the term which is an integrable function (and hence has lower singularity). Here $\tau = t - \theta$, $\tau_-(x, v) = inf\{t : (x - tv, v) \in \Gamma_-\}$, $\chi(x, y) = 1$ when the whole interval $[x, y] \subset \Omega$ and $\chi(x, y) = 0$ otherwise, $\delta$ is the Dirac delta-function in $\mathbb{R}^n$, and $\delta(;\Gamma)$ is the delta-function on $\Gamma$. Hence the albedo operator (the first singular term of its kernel) uniquely determines the integrals of $b_0$ over intersections of all straight lines with $\Omega$ and by uniqueness of the inverse $X$-ray transform $b_0$ is unique, provided it does not depend on $v$. Once $b_0$ is known the second singular term uniquely determines the collision kernel $K$.

The problem with final overdetermination (7.4.1)–(7.4.3), (7.4.5) has been considered by Prilepko and Tikhonov [PrT], [PrOV] using positive operators. Their approach is somehow similar to that initiated by the author for parabolic equations and described in some detail in Section 9.1.

Assuming that $b_0 \geq \beta > 0$, $k < \beta$ (absorption dominates collision), and $f = \rho f_1 + f_2$, where

$$\rho \in C^1([0, T]; L^2(\Omega \times W)),\ \rho \geq 0,\ \partial_t \rho \geq 0 \text{ and } \rho > 0 \text{ on } \overline{\Omega} \times \{T\} \times \overline{W},$$

they proved uniqueness of the source term $f$. Other assumptions are that $b_0$, $K$, $\rho$, and $f_2$ are given. Assuming in addition that $f \in C^1([0, T]; L_2(\Omega \times W))$, $0 \leq f$, $0 \leq \partial_t f$, and $u_T > 0$ are given functions as well as $K$, they also obtained in [PrT] uniqueness results for $b_0 = b_0(x, v)$.

In case of stationary transport equation

$$v \cdot \nabla u + b_0 u = \int_W K(\ , v, w) u(, w) dw + f \text{ in } \Omega \times W \tag{7.4.6}$$

with the boundary condition

$$u = g \text{ on } \partial\Omega \times W. \tag{7.4.7}$$

on has less data in the inverse problem and the results are weaker. The formula for the Schwarz kernel of the direct boundary value problem for stationary transport equation was given by Bondarenko [Bon] and Choulli and Stefanov [ChoS2]. In the corresponding singular terms for stationary case one only has to drop terms $\delta(\tau - \tau_-(x, v))$, $\delta(\tau - s - \tau_-(x - sv, w))$ which are due to time dependence. Accordingly, one has uniqueness of $b_0(x, |v|)$ and for $3 \leq n$ also uniqueness of $K$

As for second order elliptic equations, many boundary measurements severely overdetermine the inverse problem and is it desirable to use as few of them as needed. In case of few boundary measurements there are papers of D. Anikonov [Ani1], [Ani2], and we describe some more general results using the uniqueness theorems for the Radon transform in Sections 7.1, 7.2. In fact, there is a simple and not completely utilized relation between the two problems; in particular, one hopes that the results in inverse problems for the transport equation (7.4.6) (which is actually a differential equation of first order with parameter $v$) will be useful in integral geometry. Further on we will assume that $b_0$ does not depend on $v$.

One can write equation (7.4.6) as

$$\partial_s(e^B u(x+sv,v)) = e^B \left( \int_W K(x+sv,v,w)u(x,w)dw + f(x+sv,v) \right),$$
$$B(x+sv) = \int_0^s b_0(x+\sigma v)d\sigma.$$

Integrating and using the boundary conditions $u(x) = g(x)$, $x \in \Gamma_-$, we conclude that

$$\begin{aligned} u(x+sv,v) &= g(x,v)e^{-B(x+sv)} \\ &+ \int_0^s e^{B(x+tv)-B(x+ts)} \int_W K(x+tv,v,w)u(x+tv,w)dt \\ &+ \int_0^s e^{B(x+tv)-B(x+sv)} f(x+tv,v)dt. \end{aligned} \tag{7.4.8}$$

Using the additional boundary data (7.4.4), we conclude that we are given the right side of (7.4.8) when $x \in \partial\Omega$ and $x+sv$ is another point of $\partial\Omega$.

First, let us consider the case $n = 2$.

Let $K = 0$. Given $g_+$ for the two boundary conditions $g_- > 0$ and $g_- = 0$, one can determine $B(x+sv)$, $x+sv \in \partial\Omega$ and the weighted integral

$$\int_{L(x,y)} \rho(z,y)f(z)dl(z), \quad \rho(z,y) = \exp\left( \int_{L(z,y)} -b_0 dL \right)$$

for all directions $v$ and therefore for all straight lines $L$. From the definition of $B$ it follows that we are given the integrals of $f$ over all straight lines, so the first results on the Radon transform (Corollary 7.1.3) show that $b_0 \in L_\infty(\Omega)$ (extended as zero outside $\Omega$) is uniquely determined. Then according to next Theorem 7.4.1 the weighted integral with known weight $\rho$ obtained from the attenuation coefficient $b_0 \in C^2(\overline{\Omega})$ uniquely determines $f \in L_2(\Omega)$. When the data are available only with $g_- = 0$, we cannot say much about the simultaneous determination of $b_0$, $f$.

Now we describe results of Arbuzov, Bukhgeim, and Kasantzev [ABK] containing first global uniqueness theorems for $f \in L_2(\Omega)$ in case of arbitrary attenuation coefficient $b_0 \in C^2(\overline{\Omega})$, for some particular important collision kernel $K$. We will use some results of complex and functional analysis and use notation from section 4.4, indroducing complex variables $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ and corresponding complex differentiations $\bar{\partial}$, $\partial$. Letting $v = (cos\phi, sin\phi)$ and denoting by $u(z,\phi)$ the function $u(x,v)$ we will write the equation (7.4.6) as

$$\begin{aligned} &\bar{\partial}u(z,\phi)e^{-i\phi} + \partial u(z,\phi)e^{i\phi} + b_0(z)u(z,\phi) \\ &= \int_{-\pi}^{\pi} K(z, cos(\phi - w))u(z,w)dw + f(z),\ z \in \Omega,\ \phi \in (-\pi,\pi), \end{aligned} \tag{7.4.9}$$

where we specifieded the kernel $K$. Observe that periodicity with respect to angle and Fouirer series were used by Anikonov [Anik1]. The functions

$u(,\phi)$, $g(,\phi)$, $K(,\phi)$ are in $L_2(W)$ and $2\pi$-periodic, hence

$$u(z,\phi)=u_0(z)+2\Re\sum_{j=1}^{\infty}u_j(z)e^{-ji\phi},$$

$$g(z,\phi)=g_0(z)+2\Re\sum_{j=1}^{\infty}g_j(z)e^{-ij\phi},$$

$$K(z,\phi)=K_0(z)+\sum_{j=1}^{\infty}K_j(z)(e^{ij\phi}+e^{-ij\phi})$$

where $u_0$, $g_0$, $K_j$ are real-valued functions in $L_2(\Omega)$. Due to these expansions the equation (7.4.9) is equivalent to the equations

(7.4.10) $$\bar{\partial}u_j+\partial u_{j+2}+b_0u_{j+1}=K_{j+1}u_{j+1},\ \ j=0,1,2,\dots$$

and

$$f=2\Re\partial u_1+b_0u_0-K_0u_0.$$

Let us introduce the infinite-dimensional vectors $\mathbf{u}=(u_0,u_1,\dots)$, $\mathbf{g}=(g_0,g_1,\dots)$ in the Hilbert space $l_2$ of sequences with the standard scalar product. Let $U$ be the operator of the right shift and $U^*$ its adjoint in $l_2$:

$$U(\mathbf{u})=(0,u_0,u_1,\dots),\ \ U^*(\mathbf{u})=(u_1,u_2,\dots)$$

and the operator

$$\mathcal{K}(\mathbf{u})=(K_1u_1,K_2u_2,\dots)$$

Due to our computations, the system (7.4.10) can we written as

$$\bar{\partial}\mathbf{u}+U^*U^*\partial\mathbf{u}+b_0U^*\mathbf{u}-\mathcal{K}\mathbf{u}=0$$

or, in more convenient form,

(7.4.11) $$(\bar{\partial}-U_1\partial)\mathbf{u}+\mathcal{B}\mathbf{u}=0,\ \text{ where } U_1=-U^*U^*,\ \ \mathcal{B}=b_0U^*-\mathcal{K}$$

The boundary data (7.4.6) imply that

(7.4.12) $$\mathbf{u}=\mathbf{g}\text{ on }\partial\Omega.$$

Now we will assume that $\mathcal{K}=0$ and we will formulate one of basic results of [ABK]. The equation (7.4.11) turns out to be somehow similar to the Beltrami equation. In particular, it is shown in [ABK] there is no more than one solution to this equation satisfying the boundary condition (7.4.12) and one has an analogue of the Cauchy integral formula. To give exact formulations we need to introduce some operator functions.

Let $\omega(z,\phi)$ be the point of the intersection of $\partial\Omega$ and of the ray originated at $z\in\Omega$ with in the direction $-e^{i\phi}$, $[z_1,z_2]$ be the interval with endpoints $z_1$, $z_2$ and

$$B_0(z,\phi)=\int_{[\omega(z,\phi),z]}b_0(\zeta)|d\zeta|.$$

We define $B_{0j}(z)$ as the coefficient of the Fourier expansion

$$B_0(z, \phi) = \sum_{j=-\infty}^{\infty} B_{0,j}(z) e^{-ij\phi}.$$

and we let

$$G(z) = -2U^* \sum j = 0^\infty \bar{B}_{0,2j+1}(z)(-U_1)^j.$$

**Theorem 7.4.1.** *Let $\Omega$ be a strictly convex domain in $\mathbb{R}^2$ with the boundary of class $C^2$. Let real-valued $b_0 \in C^2(\overline{\Omega})$. Let* **u** *solve the equation (7.4.11).*

*Then*

$$\mathbf{u}(z) = 1/(2\pi i) \int_{\partial\Omega} K_{\mathcal{B}}(\zeta, z)\mathbf{u}(\zeta)(d\zeta + U_1 d\bar{\zeta}), \ z \in \Omega$$

*where $K_{\mathcal{B}}(\zeta, z) = ((\zeta - z)I + (\bar{\zeta} - \bar{z})U_1)^{-1} e^{G(z)-G(\zeta)}$ is an operator-valued function with values in $\mathcal{L}(l_{2,1}; l_{2,0})$ which is continuous with respect to $\zeta \neq z, \zeta, z \in \Omega$*

Now we give some auxiliary results explaining the solution of the problem and describing continuity properties of the operator functions $G(z)$, $K_{\mathcal{B}}(\zeta, z)$.

The first claim is that the operator function $G(z) \in C^1(\Omega; \mathcal{L}(l_{1,m})) \cap C(\overline{\Omega}; \mathcal{L}(l_{2,m})$ and it satisfies the differential equation

$$\bar{\partial} G - U_1 \partial G + b_0 U^* = 0 \text{ in } \Omega. \tag{7.4.13}$$

A proof is this claim given in [ABK], Theorem 4.2, is based on the assumptions that $\Omega$ is strictly convex and $b_0 \in C^2(\overline{\Omega})$ is real-valued. We reproduce the basic argument. Due to the convexity and regularity properties of $\Omega$ and $b_0$ we have that $\omega(z, \phi)$, $\partial\omega(z, \phi)$, $\bar{\partial}\omega(z, \phi)$ are well-defined and are continuously differentiable with respect to $\phi$. Hence the vectors

$$(B_{0,1}, B_{0,2}, \ldots), \ (\bar{\partial} B_{0,1}, \bar{\partial} B_{0,2}, \ldots)(\partial B_{0,1}, \partial B_{0,2}, \ldots)$$

are contained in $l_2$. Hence the series for and $\bar{\partial}$, $\partial$ of this series are convergent and by direct calculations it follows that the series for $G$ satisfies the differential equation (7.4.13). One of main observations in [ABK] is that the function

$$\mathbf{v}(z) = e^{-G(z)}\mathbf{u}(z)$$

solves the equation (7.4.11) with $\mathcal{B} = 0$. Loosely speaking, this transformation reduces the attenuated Radon transform to the standard nonattenuated one. As a result we have uniqueness of function $f$ entering the boundary value problem (7.4.11), (7.4.12) with $\mathcal{K} = 0$ and therefore of the function $f$ in (7.4.6), (7.4.7) with $K = 0$. Since the boundary condition (7.4.7) with $g = 0$ means that the attenuated Radon transform of the function $f$ is zero, we have uniqueness of the inversion of the attenuated Radon transform with the attenuation coefficient $b_0 \in C^2(\overline{\Omega})$ which is described in more detail below. This uniqueness and Corollary 7.1.7 combined with the standard compactness-uniqueness argument imply that for $b_0 \in C^3(\overline{\Omega})$ one can drop the last term in the Schauder type estimate of Corollary 7.1.7.

Theorem 7.4.1 is certainly valid if $f \in C^1(\overline{\Omega})$. However, uniqueness of $f \in L_1(\Omega)$ easily follows by standard argument and Corollary 7.1.7. Indeed, if $f \in L_1(\Omega)$ produces zero attenuated integrals, then smoothness of $f$ can be increased to $f \in C^1(\overline{\Omega})$ by using smoothing properties of the integral operator in (7.1.16) given for example in [Mi]. These regularity assumptions can be relaxed and a similar stability estimate can be given for Sobolev norms. Similarly, one can show that for the attenuation coefficient $b_0 \in C^3(\overline{\Omega})$ solution of the Fredholm equation (7.1.16) is unique, and hence this equation is uniquely solvable. This suggest some numerical algorithm of inversion of the attenuated Radon transform in the plane.

After the paper [ABK], Novikov gave a particular explicit inversion formula for Hölder $b_0$. His most recent results can be found in [No2]. To give an example of an inversion formulae we will use the parametrization of the attenuated Radon transform given in section 7.1, (7.1.13), letting $p = x \cdot \omega$ and

$$\rho(y, \omega) = exp(\int_0^\infty b_0(x + si\bar{\omega})ds).$$

Currently, the most general inversion formulae (in more general cases) and short and clear proofs of them are given by Boman and Strömberg [BoS]. We remind one of them claiming that

$$f(z) = 1/(4\pi)\Re(\partial((R^*_{1/\rho}\omega H R_\rho)(f)(z) \tag{7.4.14}$$

for compactly supported functions $f \in L_1(\mathbb{R}^2)$. Here $H$ is the Hilbert transform which is defined by the principal value singular integral

$$Hg(p) = 1/\pi \int_{\mathbb{R}} g(s)/(p-s)ds,$$

$\omega$ is considered as an element of $\mathbb{C}$, and as above $z = x_1 + ix_2$. Due to overdeterminancy, there are many inversion formulae for the attenuated Radon transform and advantages of one of them over another are not obvious. Probably, only possible way to select the best inversion formula is to design the most efficient numerical inversion algorithm based on this formula.

In the three-dimensional case, which is of most interest for applications of the transport equation, we have much better uniqueness results. We let $\theta$ be the angle between $v$ and the $x_3$-axis. Let $P_0$ be a half-space with its boundary parallel to the plane $\{x_3 = 0\}$ and $\Omega_0 = \Omega \cap P_0$.

**Theorem 7.4.2.** *(i) If $K = 0$ and $b_0 \in C^2(\mathbb{R}^3)$ is given, then a source term $f = f(x) \in L_\infty(\Omega_0)$ is uniquely determined in $\Omega_0$ by the function $g$ given on $\partial\Omega \cap P_0$. (ii) Assume that $\nabla_x b_0, \nabla_{x,v} K \in L_\infty(\Omega \times W \times W)$, $\nabla_{x,v} f \in L_\infty(\Omega \times W)$, and the boundary condition $g_-$ is chosen in such a way that $\|\partial_x^\alpha g_-\|_\infty(\partial\Omega \times W) < \infty$, $|\alpha| \le 1$ and $|\partial_\theta g_-| \to \infty$ when $\theta \to \pi/2$. Then the additional boundary data $g_+$ on $\partial\Omega_0 \cap P$ uniquely determine $b_0$ in $\Omega_0$.*

Part (i) follows from formula (7.4.8) and the uniqueness of the attenuated X-ray transform in the three-dimensional case (Theorem 7.1.8).

To outline a proof of (ii), we observe that from equation (7.4.6) and the boundary condition (7.4.2) one can conclude that $\nabla_x u \in L_\infty(\Omega \times W)$, and therefore on the right side of (7.4.8) all the terms, with the possible exception of the first one, have bounded derivatives with respect to $\theta$. Now we will make use of the data $g(x, \nu)$ when the plane $\{x \cdot \nu = s_0\} \cap \Omega_0$ is at positive distance from the plane $\partial P_0$; in other words, this plane must be almost parallel to $\partial P_0$. Letting the parameter $\theta$ go to $\pi/2$, we conclude that we are given the coefficient of the only singularity in the first term; i.e.,

$$B(x + s\nu) = \int_0^s b_0(x + t\nu)dt$$

when $\nu_3 = 0$ and $x \in P_0$. In other words, we are given the Radon transforms of $b_0$ in the parts of all planes $x_3 = s$ inside $P_0$. By uniqueness of the Radon transform, $b_0$ is uniquely determined.

## 7.5 Open problems

One challenging problem with important applications is related to the attenuated Radon transform.

Not very much is known about integral geometry with local data when $u(x, y)$ is given only for $x, y \in \Gamma_0$ that is a part of $\partial\Omega$. This problem is certainly fundamental for geophysics when one actually can implement only local boundary measurements, and it is quite interesting mathematically, since there is locality as in the Cauchy problem for differential equations discussed in Chapter 3.

**Problem 7.1.** Prove uniqueness of $f$ near $\Gamma_0$ when the integral (7.2.1) is given for all $x, y$ in $\Gamma_0 \subset \partial\Omega$ and when the weight function $\rho$ is monotone with respect to arc length on $L$.

In the local case $\Gamma_0 \neq \partial\Omega$ one can probably make use of weighted (Carleman-type) estimates. It seems that the global case $\Gamma_0 = \partial\Omega$ must be easier to handle, but no result is available at present. Probably, an appropriate reduction of equation (7.2.6) to a symmetric positive system in the sense of Friedrichs can be implemented and used. This problem is quite interesting even for the special attenuation function with positive variable $\mu$.

The Radon transform $F$ must satisfy consistency conditions, which are difficult to describe in the case of variable attenuation. Looking for two functions $f_1, f_2$ from the generalized Radon transform

$$\int_{y\cdot\omega=p} f_1 + \int_{y\cdot\omega>p} f_2$$

reduces the overdetermination. For available results we refer to the book of Anikonov [Anik2].

**Problem 7.2.** Find a generalization of the Radon transform with the properties of uniqueness of the inversion and with the range containing $C^\infty(S^{n-1} \times \mathbb{R})$.

Of importance for applications are the generalizations of the Radon transform onto integration manifolds of lower dimensions. Then the problem of reduction of the overdeterminacy is even more serious. In particular, very interesting is the X-ray transform

$$Pf(x, \omega) = \int_L \rho(y, L) f(y) dL(y).$$

Here $L$ is a smooth regular curve passing through the point $x$ with tangential direction $\omega$ at this point. Even the case of straight lines is challenging. When $n = 2$, we obtain the attenuated Radon transform; but when $n = 3$, the X-ray transform is quite different. If $\rho = 1$, then the obvious way to invert $P$ is to use inversion of the Radon transform in any two-dimensional plane in $\mathbb{R}^3$. If $\rho > 0$, then one can use this idea combined with "slicing" supp $f$ by planes $\pi$ such that $\pi \cap \operatorname{supp} f$ has small two-dimensional measure and then to use arguments of Corollary 7.1.7. One can even obtain uniqueness from local data described as follows. Let $E$ be a half-space in $\mathbb{R}^3$, and $\Gamma = \partial\Omega \cap E$. In the local problem, $Pf$ is given for all straight lines $L$ with the endpoints of intervals $L \cap E$ on $\Gamma$. This has been done in the paper of Isakov and Sun [IsSu1], where there are also Hölder-type stability estimates for $f$. But for curved $L$ this is an open question.

**Problem 7.3.** Consider the family of smooth curves $L$ with the property that for any $x \in \Omega$ and any direction $\omega \in S^{n-1}$ there is a unique curve $L$ through $x$ with tangential direction $\omega$ at $x$. Consider the subfamily $\mathcal{L}$ of curves $L$ of this family such that $L \cap E$ has endpoints only on $\Gamma$. Assume that for any half-space $E(x)$ such that $x \in \partial E(x)$ and $\Omega \cap E(x) \subset \Omega \cap E$ the union of curves of $\mathcal{L}$ passing through $x$ and with tangential directions in $\partial E(x)$ is contained in $E(x)$. Prove that the integrals $Pf$ over $L \in \mathcal{L}$ uniquely determine $f$ in $\Omega \cap E$.

We expect a Hölder stability estimate in this problem, too.

A different aspect of Problem 7.2 is that actually in the attenuated tomography there is hope of determining both the source and the attenuation coefficient $b_0$. Due to the apparent equivalency of this problem to the similar one for the transport equation, we formulate the question for the equation

$$v \cdot \nabla u + b_0 u = f \quad \text{in } \Omega \times W$$

with the boundary condition (7.4.2), where $g = 0$.

**Problem 7.4.** Show that with the one exception of spherical symmetry a function $g$ on $\Gamma$ uniquely determines both $v$-independent $b_0 \geq 0$ and $f > 0$ on $\Omega$.

We have noted already that inverse problems for the transport equation are largely open. In particular, we think that the following problem is of importance.

**Problem 7.5.** Show uniqueness of the absorption coefficient $b_0(x, v)$ and of the collision kernel $K(x, v, w)$, $W = \{w \in \mathbb{R}^2 : |w| = 1\}$ given the map $g_- \to g_+$ for the stationary transport equation (7.4.6)

It appears that to determine $b_0$ one can use special solutions concentrated near certain directions, which almost annihilates the collision term. After the absorption coefficient is found, one can again use solutions that are more singular near some directions to find $K$. It is not quite clear how to do this in detail. About recent results in particular cases we refer to [ChoS1], [ChoS2].

# 8

# Hyperbolic Problems

## 8.0 Introduction

In this chapter we are interested in finding coefficients of the second-order hyperbolic operator

(8.0.1) $$a_0 \partial_t^2 u + Au = f \quad \text{in } Q = \Omega \times (0, T)$$

given the initial data

(8.0.2) $$u = u_0, \partial_t u = u_1 \text{ on } \Omega \times \{0\},$$

the Neumann lateral data

(8.0.3) $$a\nu \cdot \nabla u = g_1 \qquad \text{on } \gamma_1 \times (0, T),$$

and the additional lateral data

(8.0.4) $$u = g_0 \quad \text{on } \gamma_0 \times (0, T).$$

Here $\gamma_j$ is a part of $\partial\Omega$, and $A$ is a second-order elliptic operator (4.0.1) with the (real-valued) coefficients depending on $(x, t)$. We will be interested in some cases of discontinuous coefficients $a$ or irregular lateral boundary data when a classical solution $u \in H_{(2)}(Q)$ does not exist. So we have to give appropriate definitions of generalized solutions to the initial boundary value problem (8.0.1)–(8.0.3). In particular, when $a_0 = a_0(x) \in L_\infty(\Omega)$; $a, b, c \in L_\infty(Q)$; the initial conditions $u_0 \in H_{(1)}(\Omega)$, $u_1 \in H_{(0)}(\Omega)$; and the Neumann data $g_1 \in L_2(\partial\Omega \times (0, T))$, a generalized solution $u \in H_{(1)}(Q)$ of the hyperbolic problem (8.0.1)–(8.0.3) (with $\gamma_1 = \partial\Omega$) is defined as a function satisfying the following integral relations:

(8.0.5) $$\int_Q (-a_0 \partial_t u \partial_t v + a\nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v) dQ = \int_Q f v dQ - \int_\Omega a_0 u_1 v(, 0) d\Omega + \int_{\partial\Omega \times (0,T)} g_1 v$$

for any (test) function $v \in H_{(1)}(Q)$ with $v(, T) = 0$, and the initial condition $u = u_0$ on $\Omega \times \{0\}$. In the one-dimensional case we will consider quite singular lateral Neumann data (the Dirac delta function $\delta(0)$). In this case we will define a generalized solution to (8.0.1)–(8.0.3) with $a = 1, b = 0, a_0 = a_0(x) \in C^2(\overline{\Omega})$, $c \in L_\infty(Q)$ with the zero initial data as a function $u$ that is piecewise continuous in $\overline{Q}$, continuous on $\{0\} \times [0, T]$, and that satisfies the following integral relations:

$$\int_Q u(a_0 \partial_t^2 v - \partial_x^2 v + cv) = -v(0,0) \tag{8.0.5$_1$}$$

for any function $v \in C^2(\overline{Q})$ with $v(, T) = \partial_t v(, T) = 0$ on $\Omega \times \{T\}$, $\partial_x v = 0$ on $\{0\} \times (0, T)$.

We will describe basic solvability and stability results for the direct hyperbolic problem (8.0.1)–(8.0.3)

**Theorem 8.1.** *Assume that $\partial\Omega \in C^{k+1}$ and $a_0 \in H_{k-1,\infty}(\Omega)$, $a, \partial_t a \in H_{k-1,\infty}(Q)$, $b, c \in H_{k-1,\infty}(Q)$, and $a \in H_{k,\infty}(V)$ for some neighborhood $V$ of $\partial\Omega$. Assume that the initial and Neumann data and the source term satisfy the following regularity conditions: $u_0 \in H_{(k)}(\Omega), u_1 \in H_{(k-1)}(\Omega)$, $f \in H_{(k-1)}(Q), g_1 \in H_{(k-1/2)}(\partial\Omega \times (0, T))$, and the kth-order compatibility condition on $\partial\Omega \times \{0\}$. Here $k = 1, 2, \ldots$.*

*Then there is a unique solution u to the initial boundary value problem* (8.0.1)–(8.0.3)*, and this solution satisfies the following inequality:*

$$\begin{aligned} &\|u(, t)\|_{(k)}(\Omega) + \|\partial_t u(, t)\|_{(k-1)}(\Omega) \le C(\|f\|_{(k-1)}(Q) + \|u_0\|_{(k)}(\Omega) \\ &\quad + \|u_1\|_{(k-1)}(\Omega) + \|g_1\|_{(k-1/2)}(\partial\Omega \times (0, T)). \end{aligned} \tag{8.0.6}$$

*When additionally in the case $k = 1$, $\partial_t a = 0, \partial_t b_0, \partial_t b, \partial_t c \in L_\infty(Q)$; and $u_0 \in H_{(2)}(\Omega), u_1 \in H_{(1)}(\Omega), \partial_t f \in L_2(Q), \partial_t h \in H_{(1/2)}(\partial\Omega \times (0, T))$ satisfy the second-order compatibility condition, then $\|\partial_t^2 u(, t)\|_2(\Omega)$ is bounded by the right side of* (8.0.6) *and by*

$$\|\partial_t f\|_2(Q) + \|u_0\|_{(2)}(\Omega) + \|u_1\|_{(1)}(\Omega) + \|\partial_t g_1\|_{(1/2)}(\partial\Omega \times (0, T)).$$

This result is known, and the assumptions on $g_1$ and coefficients are not optimal (optimal are not known at present, in contrast to elliptic and parabolic equations). Indeed, by subtracting from $u$ a function $w$ such that $\partial_{\nu(a)} w = g_1$ on the lateral boundary, $w$ is zero outside $V$, and $\|w\|_{(k+1)}(Q)$ is bounded by the used norm of $g_1$, we can assume that $g_1 = 0$. Then Theorem 8.1 with $k = 1$ follows from Theorems 8.1, 8.2 in the book of Lions and Magenes ([LiM], p. 265), when one chooses $V = H_{(1)}(\Omega)$, $H = L_2(\Omega)$. The additional regularity result follows from the result for $k = 1$ by differentiating the equations and the lateral boundary conditions with respect to $t$. In the differential equation for $w = \partial_t u$ we refer to the source term all terms resulting from differentiation of the coefficients $b$ and $c$. The new source term is in $L_2(Q)$, and the initial conditions are $w_0 = u_1, w_1 = -Au_0 - b_0 u_1 + f$ on $\Omega \times \{0\}$.

The compatibility conditions guarantee that $g_1$ agrees with $\partial_{\nu(a)} u_0$ on $\partial\Omega \times \{0\}$ as well as their derivatives, which can be found from equation (8.0.1) and the initial data, and which have trace on $\partial\Omega \times \{0\}$. The first-order compatibility condition does not impose any constraint. The second-order condition is $\partial_{\nu(a)}u_0 = g_1$ on $\partial\Omega \times \{0\}$. By using the Sobolev embedding theorems and Theorem 8.1, we can conclude that if $\partial\Omega$, the coefficients of the hyperbolic operator in $\overline{Q}$, and the (initial, Neumann, and right side) data are in $C^\infty(\overline{Q})$, then the solution $u \in C^\infty(\overline{Q})$. Observe that in contrast to elliptic and parabolic equations, the conditions of Theorem 8.1 are not sharp (and sharp regularity conditions are not known). Regularity results in $C^k$-spaces also follow from embedding and trace theorems. Of course, they are not sharp as well.

To introduce an important property of solutions of hyperbolic equations, we will make use of the notation $\text{con}_c(x, t) = \{(y, s) : 0 < s, |y - x| < c(t - s)\}$.

**Theorem 8.2** (Finite speed of propagation). *Let* $c$ *be* $(\sup a_0^{-1} a\xi \cdot \xi)^{1/2}$, $|\xi| = 1, (x, t) \in Q$.

*If* $f = 0$ *on* $Q \cap \text{con}_c(x, t)$, $u_0 = u_1 = 0$ *on* $\Omega \cap \text{con}_c(x, t)$, *and* $g_1 = 0$ *on* $\partial\Omega \times (0, T) \cap \text{con}_c(x, t)$, *then* $u = 0$ *on* $Q \cap \text{con}_c(x, t)$.

This result says that a solution $u$ to the mixed hyperbolic problem (8.0.1)–(8.0.3) in $Q \cap$ con depends only on traces of the data $f$, $u_0$, $u_1$, $g_1$ inside of this cone. It is well known, and we refer to the classical book of Courant and Hilbert [CouH], p, 649. Due to the finite speed of propagation, Theorem 8.2 is valid with $Q$ replaced by $Q \cap \text{con}_c(x^0, t^0)$, when the data are given on the intersections of the initial surface and the lateral wall with this cone. So generally, $\gamma_1$ can be some part of $\partial\Omega$, and then $u$ is uniquely determined and admits corresponding bounds on a subset of $Q$ that is the intersection of $Q$ with the family of cones whose intersections with $\gamma_1$ are inside this surface.

Theorems 8.1 and 8.2 and further comments guarantee that for any Neumann data $g_1 \in H_{(1/2)}(\partial\Omega \times (0, T))$ there is a unique solution $u$ of the initial boundary value problem (8.0.1)–(8.0.3) with $g_0 \in C([0, T]; H_{(1/2)}(\Omega))) \subset L_2(\partial\Omega \times (0, T))$. The last inclusion follows from the estimate (8.0.6) and the trace theorems about continuity of the trace operator from $H_{(1)}(\Omega)$ into $H_{(1/2)}(\partial\Omega)$. The claim about regularity of $g_0$ is not sharp (and there is no sharp result at present). Referring to the paper of Lasiecka and Triggiani [LaT], we observe that for smooth coefficients and boundaries the map $g_1 \to g_0$ is continuous from $L(\partial\Omega \times (0, T))$ into $H_{(1/6)}(\partial\Omega \times (0, T))$ and from $H_{(1)}(\partial\Omega \times (0, T))$ into $H_{(1)}(\partial\Omega \times (0, T))$ and is not continuous from $H_{(-1/2)}(\partial\Omega \times (0, T))$ into $H_{(1/2)}(\partial\Omega \times (0, T))$ as for elliptic equations. Most advanced results about Neumann boundary condition are given by Tataru [Tat4]. In addition, the map $g_1 \to u(; T)$ is continuous from $L_2(\partial\Omega \times (0, T))$ into $H_{(1/2)}(\Omega)$. As above, we will distinguish the cases of single and many boundary measurements when we are given the (lateral) Neumann-to-Dirichlet operator $\Lambda_l$ from $L_2(\partial\Omega \times (0, T))$ into itself. Since the Dirichlet-to Neumann map and the Neumann-to-Dirichlet operators are inverses of each other, the prescription of one of them is equivalent (at least theoretically) to the prescription of the other. We choose to consider the Neumann-to-Dirichlet operator in this chapter in part because it looks more natural in applications of hyperbolic equations. Observe that the map $g_0 \to u(, T)$ is continuous from $H_{(k)}(\partial\Omega \times (0, T))$ into $H_k(\Omega)$ at least for smooth coefficients and boundaries. This follows from classical results of Sakamoto [Sak] and standard energy integrals.

To conclude this brief introduction, we recall a known relation between hyperbolic problems and corresponding spectral problems.

Let $u_m$ be Neumann eigenfunctions of the elliptic operator $A$ when $b = 0$, $0 \le c$ and the coefficients do not depend on $t$:

$$Au_m = \lambda_m^2 u_m \text{ in } \Omega, \qquad a\nabla u_m \cdot \nu = 0 \text{ on } \partial\Omega.$$

It is known that this self-adjoint elliptic problem has an $L_2(\Omega)$ complete and orthonormal system of eigenfunctions $u_m$ corresponding to nondecreasing eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \ldots$. For simplicity we will assume that $0 < \lambda_1$. In fact, this inequality can be achieved by using substitution $u = ve^{\tau t}$. So a solution to the hyperbolic problem (8.0.1)–(8.0.3) with $a_0 = 1$, $u_0 = u_1 = 0$, $f = 0$ can be written as

$$u(x,t) = \sum g_{1m}(t) u_m(x),$$
$$g_{1m}(t) = \int_{\partial\Omega\times(0,t)} \sin(\lambda_m(t-\tau))/\lambda_m u_m(y) g_1(y,\tau)\, d\gamma(y) d\tau, \; x, y \in \partial\Omega$$

(see, e.g., the book of Lions [Li, Ch. 4, section 7, p. 320]). The series for $u$ is convergent in $L_2(\Omega)$. From the last two formulas it is not difficult to see that the Neumann-to-Dirichlet operator $\Lambda_l$ is an integral operator with the kernel

$$\sum \sin(\lambda_m(t-\tau))/\lambda_m u_m(x) u_m(y) (\text{the sum over } m = 1, 2, \ldots).$$

The data of the inverse spectral problem are Neumann eigenvalues $\lambda_m$ and the Dirichlet traces on $\partial\Omega$ the $L_2(\partial\Omega)$ normalized Neumann eigenfunctions. The above formula shows that these data uniquely determine the Neumann-to-Dirichlet lateral operator $\Lambda_l$. Therefore most of the uniqueness results for coefficients with the given $\Lambda_l$ give uniqueness results for the inverse spectral problem. For more results on inverse spectral problems we refer to [KKL].

## 8.1 The one-dimensional case

We consider two simple inverse problems that illustrate limitations and possibilities of the multidimensional situation. These problems are one-dimensional, which makes them relatively easy, but still there are some fundamental and unanswered questions about them.

Let $\Omega$ be the half-axis $(0, +\infty)$ and $Q = \Omega \times (0, T)$. Let us consider the hyperbolic equation

(8.1.1) $$(\partial_t^2 - \partial_x^2 + c)u = 0 \text{ in } Q$$

with the coefficient $c = c(x) \in C[0, \infty)$, the zero initial data (8.0.2) ($u_0 = u_1 = 0$), and the Neumann data $g_1 = \delta(0)$ on the lateral surface $\gamma_1 \times [0, T)$, $\gamma_1 = \partial\Omega$. Here $\delta$ is the Dirac delta function. When the data $g_1 \in C^1[0, T]$ and satisfy the compatibility condition $g_1(0) = 0$, there is a unique solution $u \in C^2(\overline{Q})$. Let $u_0(x, t)$

be 1 when $x < t$ and 0 when $t \le x$. This function is a generalized solution to the string equation satisfying the Neumann condition with the delta function. Let $u = u_0 + U$. Then $U$ solves the initial boundary value problem

$$(\partial_t^2 - \partial_x^2)U = -c(u_0 + U) \text{ in } Q,\ U = \partial_t U = 0 \text{ on } \Omega \times \{0\},$$
$$\partial_x U = 0 \text{ on } \partial\Omega \times (0, T).$$

We extend $U, c, \ldots$ onto $\{x < 0\}$ by the symmetric reflection $U(-x, t) = U(x, t)$, $c(-x) = c(x)$, $u_0(-x, t) = u_0(x, t)$. By d'Alembert's formula, the extended initial value problem is equivalent to the integral equation

$$U(x, t) = -\frac{1}{2}\int_{\text{con}(x,t)} c(u_0 + U), \tag{8.1.2}$$

where $\text{con}(x, t)$ is the backward characteristic triangle $\{(y, s) : |x - y| < |t - s|, s < t\}$. This Volterra-type equation has a unique solution $U \in C(\overline{Q})$. Indeed, let us define the operator

$$BU(x, t) = -\frac{1}{2}\int_{\text{con}(x,t)} cU.$$

By using induction, it is not difficult to show that

$$\|B^k U\|_\infty(Q) \le \mu^{2k} T^{2k}/(2k)!\|U\|_\infty,\ \mu = \|c\|_\infty^{1/2}(Q).$$

This bound implies that the operator $(I - B)$ is invertible (in $C(\overline{Q})$ or $L_\infty(Q)$) with inverse $(I - B)^{-1} = I + B + \cdots + B^k + \cdots$. From this formula and the bound on $\|B^k\|$ we have $\|(I - B)^{-1}\| \le e^{\mu T}$. The Volterra integral equation (8.1.2) can be written as $(I - B)U = Bu_0$, so its unique solution $U$ can be obtained via the inverse operator. From the above bounds it follows that

$$\|U(;c)\|_\infty(Q) \le \frac{1}{2}T^2 e^{\mu T}\|c\|_\infty(Q), \tag{8.1.3}$$
$$\|U(;c_2) - U(\,;c_1)\|_\infty(Q) \le \frac{1}{2}e^{\mu_1 T}(1 + 1/2\mu_2^2 T^2 e^{\mu_1 T})T^2\|c_2 - c_1\|_\infty(Q).$$

To obtain the second bound, observe that by subtracting the integral equations (8.1.2) for $U_2, U_1$ we will have

$$(U_2 - U_1)(x, t) = -\frac{1}{2}\int_{\text{con}(x,t)} (c_2 - c_1)(u_0 + U_2) - \frac{1}{2}\int_{\text{con}(x,t)} c_1(U_2 - U_1),$$

which can be considered as a Volterra integral equation with respect to $U_2 - U_1$. Denoting by $B_1$ the operator $B$ with $c$ replaced by $c_1$ as above, we obtain the bound on $\|B_1^k\|$ with $\mu$ replaced by $\mu_1 = \|c_1\|_\infty^{1/2}$, and the corresponding bound on $\|(I - B_1)^{-1}\|$. Using the first inequality (8.1.3) to bound $\|U_2\|_\infty$, we can conclude that the $L_\infty$-norm of the first term on the right side of the equation for $U_2 - U_1$ is not greater than

$$\frac{1}{2}T^2\|c_2 - c_1\|_\infty(Q)(1 + 1/2\mu_2^2 T^2 e^{\mu_1 T}).$$

Thus, we obtain the second bound (8.1.3).

Differentiating the integral (8.1.2) with respect to $x, t$, we conclude that $\nabla U \in C(\overline{Q_+})$, where $Q_+$ is $\{x \le t\} \cap Q$. The differential equation for $U$ can be understood in the generalized sense.

In the inverse problem, one is looking for $c \in C[0, +\infty)$ given the Dirichlet data (8.0.4) on $\partial\Omega \times (0, T)$.

**Lemma 8.1.1.** *The inverse problem is equivalent to the following nonlinear Volterra-type integral equation:*

$$c(\tau) + 2\int_0^\tau c(s)\partial_t U(s, 2\tau - s; c)ds = -2g_0''(2\tau),$$

$$g_0(0) = 1,\ g_0'(0) = 0,\ 2\tau < T, \tag{8.1.4}$$

*where $U$ is a solution to equation (8.1.2).*

PROOF. We define the function $v$ as 1 when $t + x < 2\tau < T$ and as 0 otherwise. We approximate $v$ by functions $v_\varepsilon(t + x)$ in $C^\infty$ such that the $v_\varepsilon$ are decreasing with respect to $\varepsilon$, and $v_\varepsilon = v$ on $\partial\Omega \times (0, 2\tau - \varepsilon)$. From the definition of a weak solution to the problem (8.1.1), (8.0.2), (8.0.3) we have

$$\int_Q u(\partial_t^2 - \partial_x^2 + c)v_\varepsilon = \int_{\partial\Omega\times(0,T)} (-g_0\partial_x v_\varepsilon - v_\varepsilon g_1)dt.$$

Using that $v_\varepsilon$ solves the homogeneous wave equation, that the $v_\varepsilon$ converge to $v$ in $L_1(Q)$ and in $L_1(\partial\Omega \times (0, T))$, and that the $\partial_x v_\varepsilon$ converge to the Dirac delta function concentrated at $2\tau$ as $\varepsilon \to 0$, in the limit we obtain

$$\int_Q cuv = -g(2\tau) - 1.$$

We denote the right side by $F_1(\tau)$ and observe that the first integral is actually over $\mathrm{Tr}(\tau)$ (the triangle with vertices at the points $(0, 0)$, $(\tau, \tau)$, $(0, 2\tau)$), where $v = 1$. Writing this integral as a multiple one, we obtain

$$\int_0^\tau c(s)\left(\int_s^{2\tau-s} (1 + U)dt\right) ds = F_1(\tau).$$

We differentiate both sides with respect to $\tau$ to get

$$\int_0^\tau c(x)2(1 + U(x, 2\tau - x))dx = F_1'(\tau).$$

Differentiating once more and using that $U(\tau, \tau) = 0$, we obtain the relation (8.1.4).

Let $c$ be any solution to equation (8.1.4) and let $g_\bullet$ be the data of the inverse problem generated by this coefficient. Starting with the relation (8.1.4) and using the conditions on $g_0$, return to the above relation containing $g_0(2\tau) + 1$. On the other hand, repeating the previous argument, we will obtain the same relation with $g_\bullet$ instead of $g_0$. Since both relations are valid when $0 < \tau < T/2$, we conclude that $g = g_\bullet$ on $(0, T)$.

The proof is complete. □

By inspecting equation (8.1.4), one concludes that when $c \in C[0, T]$, the data $g_0$ must be in $C^2[0, T]$.

**Corollary 8.1.2.** *A solution to c to the inverse problem is unique on* $(0, T/2)$.

*Let* $g_0 \in C^2[0, T]$.

*Then there is* $T_0 \leq T$ *such that a solution c to the inverse problem exists on* $[0, T_0/2]$.

PROOF. We will show that the nonlinear operator

$$B_\bullet c(\tau) = 2\int_0^\tau c(x)\partial_t U(x, 2\tau - x; c)dx$$

from $L_\infty(0, T)$ into itself admits the following bound:

$$\|B_\bullet c_2 - B_\bullet c_1\|_\infty(0, T) \leq C(M)(T - T_0)\|c_2 - c_1\|_\infty(0, T), \tag{8.1.5}$$

where $C(M) = (4Me + e^2)$, provided that $\|c_j\|_\infty \leq M^2$, $TM \leq 1, 0 \leq T_0 \leq T$, and $c_1 = c_2$ on $(0, T_0)$. This implies that for small $T - T_0$ the operator $B_\bullet$ is a contraction, and then uniqueness follows.

Observe that subtracting two integral relations (8.1.2) for $U_2$ and $U_1$, we conclude that for fixed $c_1$ the operator $U_2 - U_1$ can be considered as linear with respect to $c_2 - c_1$. Therefore, we can write the difference of values of $B_\bullet$ at $c_2$ and $c_1$ as $\mathfrak{B}(c_2 - c_1)$, where $\mathfrak{B}$ is a linear operator. We have

$$B_\bullet c_2 - B_\bullet c_1 = 2\int_0^\tau (c_2 - c_1)\partial_t U_2 + 2\int_0^\tau c_1\partial_t(U_2 - U_1).$$

Differentiating the integral (8.1.2) with respect to $t$ and subtracting the equations for $U_2$ and $U_1$, we have

$$\partial_t(U_2 - U_1) = -2^{-3/2}\int_{\partial\, \mathrm{con}(x,t)} ((c_2 - c_1)U_2 + c_1(U_2 - U_1)).$$

Using that $U_1 = 0$ when $c_1 = 0$ and the first bound (8.1.3), we conclude that $\|\partial_t U_2\|_\infty(Q) \leq M/2e^{TM}$, and therefore the $L_\infty$-norm of the first integral is bounded by

$$(T - T_0)\,\|c_2 - c_1\|_\infty Me^{TM}.$$

Using the second bound (8.1.3) with $T$ replaced by $T - T_0(c_1 = c_2$ on $(0, T_0))$, we similarly conclude that

$$\|\partial_t(U_2 - U_1)\|_\infty(Q) \leq \left(\frac{3}{2}e^{MT} + \frac{1}{2}e^{2MT}\right)(T - T_0)\|c_2 - c_1\|_\infty.$$

We conclude that $\|B_\bullet c_2 - B_\bullet c_1\|_\infty$ is bounded by

$$(4Me^{MT} + e^{2MT})(T - T_0)\|c_2 - c_1\|_\infty.$$

Using the condition $TM \leq 1$, we obtain (8.1.5).

As we noted above, global uniqueness of the solution of equation (8.1.4) follows from (8.1.5). Indeed, let $T_0$ be maximal with the property that $c_1 = c_2$ on $(0, T_0)$.

If $T_0 < T$, then we can find (probably smaller) $T$ such that $C(M)(T - T_0) < 1$. From (8.1.5) it follows that the operator $B_\bullet$ is a contraction in $L_\infty(T_0, T)$, which as in the Banach contraction theorem implies uniqueness of a solution of equation (8.1.4). So $c_1 = c_2$ on $(T_0, T)$, which contradicts the choice of $T_0$. Accordingly, $T_0 = T$, and we have global uniqueness.

Local existence also follows from the bound (8.1.5) and the Banach contraction theorem. Indeed, let $g_2(\tau) = -2g_0''(2\tau)$ and $r = \|g_2\|_\infty(0, T)$. Let $M = \sqrt{2r}$. We can assume that $T$ is so small that $TM \leq 1$ and $C(M)T < \frac{1}{2}$. Consider the set $S \subset L_\infty(0, T)$ of functions $c$ such that $\|c - g_2\|_\infty(0, T) < r$. Equation (8.1.4) can be written as $c = g_2 - B_\bullet c$. We claim that the operator on the right side of this equation maps $S$ into $S$ and is a contraction. Indeed, $\|c\|_\infty(0, T) \leq r + r = 2r$ when $c \in S$, so $\|c\|_\infty \leq M^2$, and we have the bound (8.1.5), which implies that $\|B_\bullet c_2 - B_\bullet c_1\|_\bullet \leq \frac{1}{2}\|c_2 - c_1\|_\infty$. Now it is easy to check that the operator maps $S$ into itself.

The proof is complete. □

Observe that the existence of a solution $c$ to the inverse problem is guaranteed when

$$2rT^2 \leq 1, \qquad (4\sqrt{2r}e + e^2)T < \frac{1}{2},$$

where $r = 2\|g_0''\|_\infty(0, T)$.

The next problem about determination of the speed of propagation is more complicated, even in the one-dimensional case.

We consider the equation

$$(a_0^2\partial_t^2 - \partial_x^2)u = 0 \text{ in } Q \tag{8.1.6}$$

with the same initial and lateral boundary conditions.

Again, the inverse problem is to determine $a_0$ from $g_0$. We will study it in the new (characteristic) variables

$$y = y(x) = \int_0^x a_0(\theta)d\theta, \tau = t. \tag{8.1.7}$$

**Lemma 8.1.3.** *Equation* (8.1.6) *is equivalent to the following one:*

$$\partial_\tau^2 v - \partial_y^2 v + c(y)v = 0 \textit{ in } Q, \tag{8.1.8}$$

*where*

$$c(y) = \frac{1}{2}\partial_y^2 l + \frac{1}{4}(\partial_y l)^2, l(y) = \ln a_0(x(y))$$

*for the function*

$$v(y, \tau) = a_0^{1/2}(x(y))u(x(y), \tau).$$

PROOF. This lemma follows from the following elementary calculations.

From formula (8.1.7) according to the chain rule, we have $\partial_x y = a_0(x)$, $\partial_x = a_0(x(y))\partial_y$, $\partial_t = \partial_\tau$. So

$$\partial_x u = -\frac{1}{2}a_0^{-3/2}\partial_x a_0 v + a_0^{1/2}\partial_y v,$$
$$\partial_x^2 u = a_0^{3/2}\partial_y^2 v - \left(\frac{1}{2}a_0^{-3/2}\partial_x^2 a_0 - 3/4a_0^{-5/2}(\partial_x a_0)^2\right)v,$$

and equation (8.1.6) is equivalent to equation (8.1.8) with

$$c(y) = \frac{1}{2}a_0^{-3}\partial_x^2 a_0 - \frac{3}{4}(a_0^{-2}\partial_x a_0)^2, x = x(y).$$

Observe that this expression for $c$ can be written as

$$-\frac{1}{2}a_0^{-1}\partial_x^2(a_0^{-1}) + \frac{1}{4}(\partial_x(a_0^{-1}))^2,$$

or using that according to (8.1.7), $\partial_y l = a_0^{-1}\partial_y a_0 = a_0^{-2}\partial_x a_0 = -\partial_x(a_0^{-1})$ and therefore $a_0^{-1}\partial_x^2(a_0^{-1}) = -a_0^{-1}\partial_x\partial_y l = -\partial_y^2 l$, we arrive at formula (8.1.8) for $c$.

The proof is complete. □

In the next result it is quite helpful to make use of smooth Neumann data, which are also natural in several applications. We will consider

$$g_1 \in C^k[0,T], \quad g_1^{(k-1)}(0) \neq 0, k = 1, 2, \dots. \tag{8.1.9}$$

First, we observe that due to time independence of our equations, a solution $u(;g_1)$ to equation (8.1.6) in $Q$ with the zero initial condition (8.0.2) and the Neumann condition (8.0.3) admits the following representation:

$$u(x,t;g_1) = \int_0^t g_1(s)u(x,t-s;\delta)ds. \tag{8.1.10}$$

**Exercise 8.1.4.** Prove relation (8.1.10) by using definition ($8.0.5_1$).

According to (8.1.10), the data $g_0(;g_1)$ and $g_1(;\delta)$ of the inverse problems with the Neumann data $g_1$ and $\delta$ are related via

$$g_0(t;g_1) = \int_0^t g_1(s)g_0(t-s;\delta)ds, \tag{8.1.11}$$

which can be considered a Volterra-type integral equation with respect to $g_0(;\delta)$. In addition to (8.1.9), one can assume that $g_1^{(m)}(0) = 0$ when $m < k-1$. Writing the convolution as the integral of $g_1(t-s)g_0(s)$ and differentiating equation (8.1.11) $k$ times, we arrive at the Volterra equation

$$g_0^{(k)}(t;g_1) = g_1^{(k-1)}(0)g_0(t;\delta) + \int_0^t g_1^{(k)}(s)g_0(t-s;\delta)ds,$$

which has a unique solution $g_0(t;\delta) \in C[0,T]$ given any $g_0 \in C^k[0,T]$. This implies that the Dirichlet data $g_0$ given for some Neumann data $g_1$ satisfying condition (8.1.9) uniquely determine the Dirichlet data $g_0(;\delta)$. On the other hand, the Dirichlet data for the delta function uniquely determine the general Dirichlet data via formula (8.1.11). Hence, only one set of lateral Cauchy data (with $g_1 = \delta$ or satisfying condition (8.1.9)) uniquely determines the complete lateral Neumann-to-Dirichlet map $g_1 \to g_0$.

**Lemma 8.1.5.** *The data of the inverse problem uniquely determine* $a_0(0)$, $\partial_x a_0(0)$.

PROOF. First, we observe that if $a_{01}, a_{02}$ are two coefficients producing the same data of the inverse problem, then

$$\int_Q (a_{02}^2 - a_{01}^2) v_1 v_2^* dQ = 0 \tag{8.1.12}$$

for any solution $v_1 \in H_{(2)}(Q)$ to the first equation in $Q$ satisfying the initial conditions $v_1 = \partial_t v_1 = 0$ on $(0,\infty)\times\{0\}$ and for any solution $v_2^*$ to the second equation in $Q$ satisfying the final conditions $v_2 = \partial_t v_2 = 0$ on $(0,\infty)\times\{T\}$.

To prove (8.1.12), we subtract equations (8.1.6) with the coefficients $a_{02}$ and $a_{01}$ to obtain for the difference $u$ of their solutions $u_2, u_1$ the equation

$$a_{02}^2 \partial_t^2 u - \partial_x^2 u = (a_{01}^2 - a_{02}^2)\partial_t^2 u_1 \text{ in } Q.$$

We will take as $u_{01}, u_{02}$ solutions to the hyperbolic problems (8.1.1), (8.0.2), (8.0.3) with zero initial data and with the same Neumann data $g_1 \in C^1[0,T]$, $g_1(0) = 0$. According to our assumptions and to the remark before Lemma 8.1.5, two coefficients produce the same lateral Neumann-to-Dirichlet map, so $u_{01} = u_{02}$ on $\{0\}\times(0,T)$. Therefore, the function $u$ has zero Cauchy data on this interval. Multiplying the equation for $u$ by any solution $u_2^*$ of the second equation with $u_2^* = \partial_t u_2^* = 0$ on $(0,\infty)\times\{T\}$ and integrating by parts two times on the left side of the equation and once (with respect to $t$) on the right side, we obtain the relation (8.1.12) with $\partial_t u_1$, $\partial_t u_2^*$ instead of $v_1, v_2^*$. Observe that for any $v_1, v_2^*$ satisfying the conditions in (8.1.12) the functions

$$u_1(x,t) = \int_0^t v_1(x,s)ds, \qquad u_2^*(x,t) = -\int_t^T v_2^*(x,s)ds$$

solve the same hyperbolic equations and have zero Cauchy data on $(0,\infty)\times\{0\}$, $(0,\infty)\times\{T\}$. Using these functions in the orthogonality relations obtained, we arrive at (8.1.12).

To complete the proof, assume the opposite. Then there are two coefficients $a_{01}, a_{02}$ with $a_{01}(0) < a_{02}(0)$ or with $a_{01}(0) = a_{02}(0)$ and $\partial_x a_{01}(0) < \partial_x a_{02}(0)$. In cases $0 < (a_{02} - a_{01})$ on the interval $(0,\varepsilon)$ for some positive $\varepsilon$. Let $v_1$ be the solution to the first equation in $Q^* = \mathbb{R}\times(0,T)$ with right side $f \in C^1(\overline{Q^*})$ that is nonnegative and with supp $f = \{x \le 0\}\times[0,T_0]$, and with zero Cauchy data on $\mathbb{R}\times\{0\}$. Let $v_2^*$ be the solution of the second equation with the same right side and with zero Cauchy data on $\mathbb{R}\times\{T\}$. By using Riemann's function [CouH, p. 453], it is not difficult to show that $v_1 > 0$ near the origin above the characteristic $t = \gamma_1(x)$ passing through $(0,0)(\gamma_1'(x) = a_{01}^{-1}(x), \gamma_1(0) = 0)$ and that $v_2^* > 0$ near the origin below the characteristic $t = \gamma_2(x)$ passing through the point $(0, T_0)(\gamma_2'(x) = -a_{02}^{-1}(x), \gamma_2(0) = T_0)$ when $T_0$ is sufficiently small. For such $v_1, v_2^*$ the integral (8.1.12) is negative. The contradiction shows that our assumption was wrong.

The proof is complete. □

To formulate uniqueness results for the coefficient, we assume that $a_0 \leq a^\bullet$.

**Corollary 8.1.6.** *Let* $\Omega = (0, 1)$. *The coefficient* $a_0(x) \in C^2([0, +\infty))$ *of the hyperbolic equation* (8.1.6), *with zero initial conditions and the lateral Neumann condition* $-\partial_x u = \delta$ *or* $-\partial_x u = g_1$ *on* $\{0\} \times [0, T]$, *where* $g_1$ *satisfies conditions* (8.1.9), *is uniquely determined on* $[0, T/(2a^\bullet)]$ *by the additional Dirichlet data* $u = g_0$ *on* $\{0\} \times [0, T]$.

PROOF. First, we consider the Dirac function as the Neumann data. From Lemmas 8.1.3, 8.1.5 and Corollary 8.1.2 we conclude that the coefficient $c(y)$ given in (8.1.8) is uniquely determined on $(0, T/2)$ by the data of the inverse problem. By Lemma 8.1.5 the Cauchy data $l(0), \partial_y l(0)$ are uniquely determined as well. Solving the Cauchy problem for the ordinary differential equation (8.1.8) with respect to $l$, we obtain uniqueness of $l(y)$ when $0 < y < T/2$. So we are given $a_0(x(y)) = e^{l(y)}$. From the definition (8.1.7) of the substitution $y = y(x)$, we have $dx/dy = a_0^{-1}(x(y)) = e^{l(y)}$, and therefore

$$x(y) = \int_0^y e^{-l(s)} ds, 0 < y < T/2.$$

Since $a_0 < a^\bullet$, the inverse function $y(x)$ is uniquely determined, at least on the interval $0 < x < T/(2a^\bullet)$. Finally, we have $a_0(x) = e^{l(y(x))}$ on this interval.

The claim about the Neumann data $g_1$ satisfying condition (8.1.9) follows from the observation that the Dirichlet data $g_0$ for $g_1$ uniquely determine the Dirichlet data for the delta function.

The proof is complete. □

**Corollary 8.1.7.** *(i) The coefficient* $c \in C[0, 1]$ *of equation* (8.1.1) *with zero initial conditions* (8.0.2) *and Neumann condition* (8.0.3) *on* $\{0\} \times (0, T)$ *with* $g_1 = \delta$ *or satisfying condition* (8.1.9) *and zero on* $\{1\} \times (0, T)$ *is uniquely determined on* $[0, T/2]$, $T < 2$, *by the additional Dirichlet data* $u = g_0$ *on* $\{0\} \times [0, T]$.

*(ii) The coefficient* $a_0 \in C^2[0, 2]$ *of equation* (8.1.6) *with the same initial and lateral boundary conditions as in (i) is uniquely determined on* $[0, T/(2a^\bullet)]$, $T < 2a^\bullet$, *by the additional Dirichlet data on* $\{0\} \times (0, T)$.

PROOF. (i) First, we extend $u$ and $c$ as even functions with respect to $(x - 1)$ and use the same notation for the extended functions. From the zero Neumann boundary condition on $\{1\} \times (0, T)$, we will obtain the same hyperbolic equation in the extended domain $(0, 2) \times (0, T)$. Since in our problem the speed of propagation is 1 (Theorem 8.2) and the initial data are zero, its solution coincides on con$(0, T)$ with the solution of the problem in the strip $(0, \infty) \times (0, T)$. Now Corollary 8.1.7(i) follows from Corollary 8.1.2.

(ii) By extending $u$ and $c$ onto $(0, 2)$ as above the using that the speed of propagation does not exceed $1/(2a^\bullet)$, we can extend $u$ onto $\mathbb{R}_+ \times (0, T) \cap \{t < 2a^\bullet x\}$ as zero, preserving the equation and the initial and lateral boundary data on $\{0\} \times (0, T)$. After this preparation, Corollary 8.1.7(ii) follows from Corollary 8.1.6.

The proof is complete. □

**Exercise 8.1.8.** By using the substitutions (8.1.7) and $u = wv$ with an appropriate function $w = w(x)$, show that the general second-order hyperbolic equation

$$a_0^2 \partial_t^2 u - \partial_x^2 u + b_0 \partial_t u + b \partial_x u + cu = 0$$

with time-independent coefficients $a_0, b_0, b$, and $c$ can be transformed into the simpler equation

$$\partial_t^2 v - \partial_x^2 v + B \partial_t v + C v = 0, \tag{8.1.13}$$

also with time-independent coefficients $B, C$.

**Exercise 8.1.9.** Consider equation (8.1.1) with $c = c(x, t) \in L_\infty(Q)$. Show that the lateral Neumann-to-Dirichlet map $g_1 \to g_0$ for this equation uniquely determines $c$ on the triangle with vertices $(0, 0)$, $(0, T)$, $(T/2, T/2)$.

{*Hint*: Repeating the argument of Lemma 8.2.1, derive for $c(x, t)$ the nonlinear Volterra-type integral equation

$$c(\kappa, \tau) + (\partial_\tau - \partial_\kappa) \int_0^\kappa cU(x, \tau + \kappa - x; \tau - \kappa) dx = (\partial_\tau^2 - \partial_\kappa^2) g(\tau + k; \tau - \kappa).$$

Here $g_0(; \tau) = u(0, ; \tau)$, where $u(; \tau)$ is the solution to the hyperbolic problem with the Neumann data $\delta(-\tau)$ (and zero initial data). Use that due to the finite speed of propagation $u(x, t; \tau) = 0$ when $t - x < \tau$.}

For one-dimensional inverse problems and their applications we refer to the book of Gladwell [Gl]. For one-dimensional inverse spectral problem we refer to Pöschel and Trubowitz [PoT].

## 8.2 Single boundary measurements

As for elliptic equations, one can prescribe one set of the initial and boundary data (8.0.2), (8.0.3) and then try to recover, say, the coefficient $a(x)$ from observation (8.0.4). Since the problem is local in time and space (due to finite speed of propagation), it is reasonable to consider small times and replace the original nonlinear inverse problem by its linearization (for $\Omega$ that is the strip $\{0 \le x_n \le T\}$, and around $b_0$ that depends only on $x_n$). As an example of available results we will formulate Theorem 8.2.1 where we consider equation (8.0.1) with $a_0 = 1, a = 1, b = -\rho^{-1} \nabla \rho, c = 0$, which is a form of the acoustic equation $\rho \partial_t^2 u - \operatorname{div}(\rho \nabla u) = 0$. The linearization of the inverse problem around $\rho_0$

($\rho = \rho_0 + f$ with "small" $f$) consists in finding $(v, f)$ entering the equations

$$\partial_t^2 v - \Delta v - \nabla \log \rho_0 \cdot \nabla v$$
$$= \nabla(\rho_0^{-1} f) \cdot \nabla u(;0) \text{ in } \{(x,t) : x_n < t, 0 < t < 2T\},$$

$$\partial_n v = 0 \text{ on } \gamma \times (0, 2T),$$
$$v(x', t, t) = -\frac{1}{2}\rho_0(t)^{-3/2} f(x', t) \text{ as } x \in \gamma, 0 < t < T,$$
$$v = g_0 \text{ on } \gamma \times (0, 2T),$$

where $\gamma$ is $\{x_n = 0\}$, $g_0$ is a given function, and $u(;0)$ is a solution to the initial boundary value problem (8.0.1), (8.0.2), (8.0.3) with $\rho = \rho_0, u_0 = u_1 = 0$, and $g_1 = \delta(t)$ (Dirac delta-function). It is not difficult to observe that $u(;0)$ depends only on $x_n$ and $t$.

We give the following result of Sacks and Symes [SSy].

**Theorem 8.2.1** (Uniqueness for Linearization). *Assume that* $\log \rho_0 \in H_{\infty,1}(0, T)$, $\rho_0 > \varepsilon > 0$, *and that* $f \in H_{1,1}(\Omega)$, $f = 0$ *on* $\overline{\Omega} \cap \{x_n = 0\}$.

*If* $g_0 = 0$, *then* $f = 0$ *on* $\Omega$.

We will not give details of a proof, observing only that by using the Fourier transform with respect to $x'$ one can reduce the multidimensional (linear) hyperbolic problem to a one-dimensional problem depending on the Fourier variable as a parameter and then make use of Volterra-type integral equations as in Section 8.1. The Fourier transform will transform the differential equation into a differential equation because $b_0$ and $u(;0)$ do not depend on $x'$.

Linearization makes certain sense for small $T$ due to the finite speed of propagation. However, there are no proofs that the range of the operator that maps $f$ into $g_0$ is closed in standard functional spaces, so one cannot apply contraction arguments and derive from Theorem 8.2.1 uniqueness results for the original nonlinear inverse problem. Even for the linearization there are no results when the data are given on a part of $\partial\Omega$ (local data), which is a typical situation in geophysics. The situation is more complicated when the speed of propagation is variable. Linearization is also discussed by Romanov [Rom], p. 134, where there are uniqueness results for $c$ obtained by using integral geometry over a family of ellipsoids.

Now we will discuss the original (nonlinear) inverse problem. There are global uniqueness results under a quite restrictive condition on the initial data. In particular they can not be zero. The method of proof is based on Carleman estimates. It was orginated by Bukhgeim and Klibanov [BuK] and later modified by Imanuvilov and Yamamoto [IY2]. By using their ideas we will give a general result applicable to systems of equations of second order (including elasticity system and systems with principal parts which correspond to parabolic and Schrödinger equations). Let $\mathbf{A}$ be a $m \times m$-matrix partial differential operator of second order in a domain $Q \subset \mathbb{R}^{n+1}$ with time independent $L_\infty(\Omega)$-coefficients. We let $x = (x_1, \ldots, x_n)$, $t = x_{n+1}$. Let us consider the following inverse source problem

(8.2.1) $$\mathbf{Au} = \mathcal{A}\mathbf{f},\ \partial_t \mathbf{f} = \mathbf{0} \text{ in } Q = \Omega \times (-T, T)$$

with a weight $m \times m$ matrix- function $\mathcal{A}$. Here the vector-functions $\mathbf{u} = (u_1, \dots, u_m)$ and $\mathbf{f} = (f_1, \dots, f_m)$ are to be found from additional initial and lateral boundary data.

We introduce a Carleman type estimate

(8.2.2) $$\int_Q e^{2\tau\varphi}(\tau^2|\mathbf{u}|^2 + m\tau|\partial_t \mathbf{u}|^2) \leq C \int_Q e^{2\tau\varphi}|\mathbf{Au}|^2$$

for all functions $\mathbf{u} \in C_0^2(Q)$ and $C \leq \tau$. Here $m = 0, 1$, $\varphi$ is a $C^2(\overline{Q})$-function with nonvanishing gradient on $\overline{Q}$. As in chapter 3 we let $Q_\varepsilon = Q \cap \{\varepsilon < \varphi\}$. We will consider two cases: 1) $m = 0$, and 2) $m = 1$ or $\mathbf{A}$ does not involve $\partial_t \partial_j$, $j = 1, \dots, n+1$. Case 1) is applicable to the isotropic elasticity ( see (3.5.11)). Case 2) corresponds to standard Carleman estimates for scalar hyperbolic operators or to parabolic or Schrödinger operators (see Theorem 3.2.1, proof of Theorem 3.3.10, Theorem 3.3.12, and proof of Theorem 3.4.13).

**Theorem 8.2.2.** *Let $\mathcal{A}, \dots, \partial_t^2 \mathcal{A} \in C(\overline{Q})$ and in case 1) let $\partial_t^3 \mathcal{A} \in C(\overline{Q})$. Let the weight function $\varphi$ satisfy conditions*

(8.2.3)
$$\varphi < 0 \text{ on } \partial Q \setminus \Gamma,\ \Gamma \subset \partial\Omega \times (-T, T),\ \varphi(,t) \leq \varphi(,0),\ \text{when } -T < t < T.$$

*Let $\mathbf{A}$ does not involve differentiations $\partial_t \partial_k$, $k = 1, \dots, n$ and the Carleman estimate (8.2.2) be valid. Let $\partial_t^j \partial^\alpha \mathbf{u} \in L_2(Q)$, $j = 0, \dots, 3$, $|\alpha| \leq 1$ and $\mathbf{f} \in L_2(\Omega)$. Let*

(8.2.4) $$det(\mathcal{A}) \neq 0 \text{ on } Q_0 \cap \{t = 0\}$$

*Then the equalities*

(8.2.5) $$\mathbf{u} = \mathbf{0} \text{ on } \Omega \times \{0\}$$

*and*

(8.2.6) $$\mathbf{u} = \partial_\nu \mathbf{u} = \mathbf{0} \text{ on } \Gamma$$

*imply that $\mathbf{f} = \mathbf{0}$ on $Q_0$.*

PROOF. We consider case 1). Differentiating the equations (8.2.1) with respect to $t$ and using time independence of coefficients of $\mathbf{A}$ we obtain

$$\mathbf{A}\partial_t \mathbf{u} = \partial_t \mathcal{A}\mathbf{f},\ \mathbf{A}\partial_t^2 \mathbf{u} = \partial_t^2 \mathcal{A}\mathbf{f},\ \mathbf{A}\partial_t^3 \mathbf{u} = \partial_t^3 \mathcal{A}\mathbf{f} \text{ in } Q.$$

Let $\varepsilon > 0$. Let $\chi$ be a $C^\infty(\mathbb{R}^{n+1})$-function, $\chi = 1$ on $Q_\varepsilon$ and $\chi = 0$ on $Q \setminus Q_0$. Since $\mathbf{u}$ has zero Cauchy data on $\Gamma$ the functions $\mathbf{A}\partial_t^j(\chi\mathbf{u})$ can be $L_2(Q)$ approximated by $C_0^2(Q)$-functions, so for them we have the Carleman estimate (8.2.2). From the Leibniz' formula we have $\mathbf{A}(\chi\mathbf{u}) = \chi\mathbf{Au} + \dots$ where $|\dots| \leq C|\nabla\mathbf{u}|$. Using that $\chi = 1$ on $Q_0$ and shrinking the integration domain in the left side of (8.2.2) for

$\chi\mathbf{u}, \ldots \partial_t^3(\chi\mathbf{u})$ we obtain

$$\int_{Q_\varepsilon} e^{2\tau\varphi} \sum_{j=0,\ldots,3} \tau^2|\partial_t^j\mathbf{u}|^2 \leq C(\int_Q e^{2\tau\varphi}|\mathbf{f}|^2 + \int_{Q\setminus Q_\varepsilon} e^{2\tau\varphi}\sum|\partial_t^j\partial^\alpha\mathbf{u}|^2)$$

$$\leq C(\int_Q e^{2\tau\varphi}|\mathbf{f}|^2 + e^{2\tau\varepsilon}). \tag{8.2.6}$$

where the sums are over $j = 0, \ldots, 3$, $|\alpha| \leq 1$ and we used that $\varphi \leq \varepsilon$ on $Q\setminus Q_\varepsilon$. Here and further in the proof $C$ denote generic constants depending on $Q, \mathbf{u}, \varphi$ but not on $\tau$.

On the other hand, from the equation (8.2.1) at $t = 0$ and from the initial condition (8.2.5) we get $\mathbf{A}_2\partial_t^2\mathbf{u}(,0) = \mathcal{A}(,0)\mathbf{f}$ on $\Omega$ where $\mathbf{A}_2$ is a $L_\infty(\Omega)$-matrix function. Using condition (8.2.4) we yield $|\mathbf{f}| \leq C|\partial_t^2\mathbf{u}(,0)|$. According to condition (8.2.3)

$$\int_Q e^{2\tau\varphi}|\mathbf{f}|^2 \leq 2T\int_\Omega e^{2\tau\varphi(,0)}|\partial_t^2\mathbf{u}|^2(,0). \tag{8.2.7}$$

On the other hand,

$$\begin{aligned}\int_\Omega e^{2\tau\varphi(,0)}|\partial_t^2\mathbf{u}|^2(,0) &= -\int_\Omega\int_0^T \partial_s(e^{2\tau\varphi(,s)}|\partial_t^2\mathbf{u}(,s)|^2)ds \\ &\quad + \int_\Omega e^{2\tau\varphi(,T)}|\partial_t^2\mathbf{u}(,T)|^2 \\ &\leq C(\int_Q e^{2\tau\varphi}(\tau|\partial_t^2\mathbf{u}|^2 + |\partial_t^3\mathbf{u}|^2) + 1)\end{aligned}$$

where we used that $\varphi(,T) \leq 0$. Combining this inequality with (8.2.6) and (8.2.7) we arrive at

$$\tau^2\int_{Q_\varepsilon} e^{2\tau\varphi}\sum_{j=0,\ldots,3}|\partial_t^j\mathbf{u}|^2 \leq C(\tau\int_Q e^{2\tau\varphi}\sum_{j=0,\ldots,3}|\partial_t^j\mathbf{u}|^2 + e^{2\tau\varepsilon}).$$

Splitting $Q$ in the right side into $Q_\varepsilon$ and $Q\setminus Q_\varepsilon$, choosing $\tau$ large to absorb the integral over $Q_\varepsilon$ by the left side and observing that the remaining integral is less than $C\tau$ we finally yield

$$\tau^2\int_{Q_\varepsilon} e^{2\tau\varphi}|\mathbf{u}|^2 \leq C(\tau+1)e^{2\tau\varepsilon}$$

Dividing both sides by $\tau$, replacing $\varphi$ on $Q_\varepsilon$ by its smallest value $\varepsilon$ and letting $\tau \to \infty$ we conclude that $\mathbf{u} = \mathbf{0}$ on $Q_\varepsilon$. Since $\varepsilon$ is any positive number, $\mathbf{u} = \mathbf{0}$ on $Q_0$ and from (8.2.4) we obtain the conclusion of Theorem 8.2.2.

The proof in case 2) is similar. Only difference is that it is sufficient to differentiate the equation (8.2.1) with respect to $t$ twice. □

Observe that conditions of this Theorem can be slightly relaxed, at expense of longer proofs. For example, regularity of solution $\mathbf{u}$ can be raised in $Q_0$ by using Carleman estimates. Condition (8.2.4) however is hard to remove.

As an application let us consider the initial problem (8.0.1)–(8.0.3) with $a_0 = 1, A = -a\Delta, a \in C^l(\overline{\Omega}), l > (n+7)/2, g_1 \in C^l(\partial\Omega \times [0, T]), u_0 \in C^l(\overline{\Omega})$, and $\partial\Omega \in C^l$. We will consider cases (3.4.2), (3.4.3) where $G$ is replaced by $\Omega$. We assume that $\gamma_1 = \partial\Omega$ and $\gamma_0 = \gamma$ from (3.4.2), (3.4.3). We let $Q_\varepsilon = Q \cap \{\varepsilon < x_1^2 + \cdots + x_{n-1}^2 + (x_n - \beta_n)^2 - \theta^2 t^2 - s\}$, where $\beta_n$ and $s$ are chosen as in Theorem 3.4.1. We assume also the compatibility condition on $\partial\Omega \times \{0\}$ of order $l$. These conditions are certainly satisfied when $a = 1$ near $\partial\Omega$ and $g_1 = \partial_\nu u_0$ on $\partial\Omega$, where $u_0$ is any smooth function on $\mathbb{R}^n$ with $-\Delta u_0 > 0$ on $\overline{\Omega}$. Let $u(;1), u(;2)$ be solutions to the hyperbolic initial boundary value problem (8.0.0)–(8.0.3) (with $A = -a_1\Delta, -a_2\Delta, a_0 = 1, f = 0, b = 0, c = 0$).

**Corollary 8.2.3.** *Assume that the numbers $\theta, T$ and the functions $a_1^{-1/2}, a_2^{-1/2}$ satisfy conditions* (3.4.5) *and that the initial data satisfy the conditions* $0 < -\Delta u_0$, $0 < u_0, u_0 \in C^l(\overline{\Omega}), u_1 = 0$ *on* $\Omega$.

*If* $u(;1) = u(;2)$ *on* $\gamma \times (0, T)$, *then* $a_1 = a_2$ *on* $\Omega_0$.

PROOF. First, we extend $u(;j)$ onto negative $t$ by letting $u(x, -t; j) = u(x, t; j)$. Then the differential equations will hold in the extended domain $Q_0$, which we will denote by the same symbol. Subtract the equations for $u(;2)$ and $u(;1)$ and let $u = u(;2) - u(;1)$, $f = a_2 - a_1$. Then we will have the following equations:

$$\partial_t^2 u - a_2\Delta u = \Delta u(;1) f, \ \ \partial_t f = 0 \text{ on } Q,$$
$$u = 0 \ \text{ on } \ \Omega \times \{0\}, \ \ u = \partial_\nu u = 0 \text{ on } \gamma \times (0, T).$$

As shown in the proof of Theorem 3.4.1, the function $\psi(x, t) = x_1^2 + \cdots + x_{n-1}^2 + (x_n - \beta_n)^2 - \theta^2 t^2 - s$ is pseudo-convex with respect to the hyperbolic operator $A = \partial_t^2 - a_2\Delta$, hence the function $\varphi = e^{\lambda\psi} - 1$ is strongly pseudo-convex for some large $\lambda$, as demonstrated in the proof of Theorem 3.2.1', and we have the Carleman estimate (8.2.2). Conditions (8.2.3) follow from (3.4.5) and from the choice of $\psi$. By Theorem 8.2.2 $f = 0$ on $Q_0$.

The proof is complete. □

Conditions on the initial data are not quite suitable for many applied problems, but Theorem 8.2.2 (and corollaries for elliptic, parabolic, and hyperbolic equations) provide the only uniqueness results available for many nonoverdetermined inverse coefficients problems. In [Is4] there are also results about the simultaneous determination of two coefficients $a_0$ and $c$ from two (independent) lateral boundary measurements.

Stability estimates in this Corollary were given by Khaidarov [Kh2]. They are similar to the estimate (3.2.6), but they are more complicated.

When the Dirichlet (or Neumann data) are given on the whole of $\partial\Omega$ and the additional lateral boundary data on a "large" part $\Gamma_0$ of $\partial\Omega$ as in Theorem 3.4.8, one can expect much better stability. Indeed, Imanuvilov and and Yamamoto [IY2] obtained the followng conditional Lipschitz stability estimate.

Let $u(;2)$, $u(;1)$ be solutions of the hyperbolic initial boundary value problem

$$\partial_t^2 u(;j) - \Delta u(;j) + c_j u(;j) = 0 \text{ in } Q,$$

$$\partial_\nu u(;)j = 0 \text{ on } \partial\Omega \times (0, T),$$

and with the initial data $u_0 \in H_{(3)}(\Omega)$, $\partial_\nu u_0 = 0$ on $\partial\Omega \times (0, T)$, $u_1 = 0$.

**Theorem 8.2.4.** *Let $diam\ \Omega < T$. Let $\varepsilon_0 < u_0$ on $\Omega$.*

*Then there is a constant $C$ depending only on $\Omega, T, u_0, \|c_j\|_{1,\infty}(\Omega)$ such that*

$$\|c_2 - c_1\|_2(\Omega) \le C\|\partial_t(u(;2) - u(;1))\|_{(1)}(\partial\Omega \times (0, T)).$$

The proofs in [IY2] are based on the linearization which reduces the coefficient problem to the inverse source problem, like in Theorem 8.2.2 and Corollary 8.2.3, and on Carleman estimates.

The even reflection with respect to $t$ which was used in the proof of Corollary 8.2.3 can not be applied to the equation $\partial_t^2 u - \Delta u + b_0 \partial_t u = 0$ and a for some time uniqueness of damping coefficient $b_0(x)$ from the same data as in Corollary 8.2.3 was not known and was posed as Problem 8.3 in the first edition of this book. If $b_0 \neq 0$, then this reflecton defines solutions of the wave equation with time discontinuous coefficient $b_0$ and hence the known uniqueness proof could not be applied. Bukhgeim, Cheng, Isakov, and Yamamoto [BuCIY] combined singular behavior of solutions at $t = 0$ with the initial condition and showed that Carleman estimates method still leads to uniqueness result quite similar to Corollary 8.2.3. provided condition $\varepsilon_0 < u_0$ is replaced by condition $\varepsilon_0 < u_1$.

For the acoustic equation $\partial_t^2 u - \mathrm{div}(a\nabla u) = 0$ one can not apply Theorem 8.2.2 because the source term is now a differential operator $div(f\nabla u_1)$. Imanuvilov and Yamamoto [IY3] modified the scheme of the proof of Theorem 8.2.2 by using Carleman estimates in the Sobolev space $H_{(-1)}(Q)$ of negative order, so one can handle the source term like in Theorem 8.2.2. In [IY3] there are global uniqueness and Hölder stability results for $a$ similar to Corollary 8.2.3.

The same idea was applied by Imanuvilov, Isakov and Yamamoto [IIY] to the far more difficult case of linear isotropic elasticity system

$$\rho(x)\partial_t^2 \mathbf{u} - \mu(x)(\Delta \mathbf{u} + \nabla\,\mathrm{div}\,\mathbf{u}) - \nabla(\lambda(x)\,\mathrm{div}\,\mathbf{u}) -$$

(8.2.8)
$$\sum_{j=1}^{3} \nabla\mu \cdot (\nabla u_j + \partial_j \mathbf{u})\mathbf{e}_j \quad \text{in} \quad Q = \Omega \times (-T, T)$$

for the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$. Here $\Omega$ is a domain in $\mathbb{R}^3$ with $C^4$-boundary and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis in $\mathbf{R}^3$. We let $d = inf|x|$ and $D = sup|x|$ over $x \in \Omega$. By translating $\Omega$ we can assume that $D^2 < 2d^2$. As above we introduce $Q(\varepsilon) = Q \cap \{\varepsilon < |x|^2 - \theta^2 t^2 - d^2\}$. Let us indroduce conditions

$$-\theta_0 < x \cdot \nabla a/a, \ a^2\theta^2 + 2|\nabla a|\theta d < 1 - \theta_0 \text{ on } \Omega \tag{8.2.9}$$

with some positive $\theta_0$, and the class of the elastic parameters

$$\mathcal{E} = \mathcal{E}_{\theta,\theta_0,\varepsilon_0,M} = \{(\lambda, \mu, \rho) : |\lambda|_6(\bar{\Omega}) + |\mu|_7(\bar{\Omega}) + |\rho|_7(\bar{\Omega}) \\ \leq M, \varepsilon_0 < \lambda + \mu, \varepsilon_0 < \mu, \varepsilon_0 < \rho \text{ on } \bar{\Omega},$$

$$a = (\rho/\mu)^{1/2} \text{ and } a = (\rho/(2\mu + \lambda))^{1/2} \text{ satisfy (8.2.9)}\}$$

where $\varepsilon_0 > 0$ and $M > 0$ are some positive constants. We will use two sets $\mathbf{G}(; j)$, $j = 1, 2$ of the initial and lateral boundary data $\mathbf{u}_0 \in H_{(8)}(\Omega), \mathbf{u}_1 \in H_{(7)}(\Omega)$, $\mathbf{g} \in C^7([-T, T]; H_{(1)}(\partial\Omega))$ satisfying the standard compatibility conditions at $\partial\Omega \times \{0\}$ (of order 7). Differentiating the equations (8.2.8) and the lateral boundary conditions with respect to $t$ and using known energy estimates for the elasticity system [Ci] and Sobolev imbedding theorems we conclude that

$$|\partial_t^6 \mathbf{u}(; \rho, \lambda, \mu, \mathbf{G})|_0(\overline{Q}) + |\partial_x^\alpha \partial_t^5 \mathbf{u}(; \rho, \lambda, \mu, \mathbf{G})|_0(\overline{Q}) < C \text{ when } |\alpha| \leq 1,$$

for all elastic parameters in $\mathcal{E}$ and the initial and boundary data $\mathcal{G}$ of the described regularity.

To guarantee uniqueness of elastic parameters we will use two sets $\mathbf{G}(; j)$ of the initial and lateral boundary data $(\mathbf{u}_0(; j), \mathbf{u}_1(; j), \mathbf{g}(; j))$, $j = 1, 2$. Denote by $\mathbf{D}$ the $12 \times 7$-matrix

$$\begin{pmatrix} \mu_1 \Delta \mathbf{u}_0(; 1) + (\lambda_1 + \mu_1)\nabla(div\mathbf{u}_0(; 1)) & (div\mathbf{u}_0(; 1))\mathbf{I}_3 & \nabla\mathbf{u}_0(; 1) + (\nabla\mathbf{u}_0(; 1))^T \\ \mu_1 \Delta \mathbf{u}_1(; 1) + (\lambda_1 + \mu_1)\nabla(div\mathbf{u}_1(; 1)) & (div\mathbf{u}_1(; 1))\mathbf{I}_3 & \nabla\mathbf{u}_1(; 1) + (\nabla\mathbf{u}_1(; 1))^T \\ \mu_1 \Delta \mathbf{u}_0(; 2) + (\lambda_1 + \mu_1)\nabla(div\mathbf{u}_0(; 2)) & (div\mathbf{u}_0(; 2))\mathbf{I}_3 & \nabla\mathbf{u}_0(; 2) + (\nabla\mathbf{u}_0(; 2))^T \\ \mu_1 \Delta \mathbf{u}_1(; 2) + (\lambda_1 + \mu_1)\nabla(div\mathbf{u}_1(; 2)) & (div\mathbf{u}_1(; 2))\mathbf{I}_3 & \nabla\mathbf{u}_1(; 2) + (\nabla\mathbf{u}_1(; 2))^T \end{pmatrix}$$

constructed from these data.

Let $\omega$ be a subdomain of $\Omega$ with $\partial\Omega \subset \partial\omega$ (boundary layer) and $Q_\omega = \omega \times (-T, T)$.

**Theorem 8.2.5.** *Let us assume that* $(\rho, \lambda, \mu), (\rho_1, \lambda_1, \mu_1)$ *are in* $\mathcal{E}_{\theta,\theta_0,\epsilon_0,M}$, *that* $\lambda_1 = \lambda, \mu_1 = \mu$ *on* $\partial\Omega$ *and that for some* $\varepsilon_0 > 0$ *we have:*

*at any point of* $\bar{\Omega}$ *absolute value of one of*

(8.2.10) $7 \times 7$ *minors of the matrix* $\mathbf{D}$ *is not less than the number* $\varepsilon_0$.

*Then there are constants* $C$ *and* $\kappa \in (0, 1)$ *depending only on* $Q, \mathcal{E}, \mathbf{G}, \theta, \varepsilon_0$ *such that*

$$\|\lambda_1 - \lambda\|_2(\Omega) + \|\mu_1 - \mu\|_2(\Omega) + \|\rho_1 - \rho\|_2(\Omega)$$

$$\leq C \sum_{j=1}^{2} \|\mathbf{u}(; \lambda_1, \mu_1, \rho_1; \mathbf{G}(; j)) - \mathbf{u}(; \lambda, \mu, \rho; \mathbf{G}(; j))\|_{(5)}^{\kappa}(Q_\omega)$$

*for all* $(\rho, \lambda, \mu), (\rho_1, \lambda_1, \mu_1) \in \mathcal{E}_{\theta,\theta_0,\varepsilon_0,M}$.

The condition (8.2.10) is somehow restrictive, but it guarantees the independency of the data $(\mathbf{u}_0(;1), \mathbf{u}_1(;1))$ and $(\mathbf{u}_0(;2), \mathbf{u}_1(;2))$. For example, it is satisfied for

$$\mathbf{u}_0(x;1) = (x_1x_2, 0, 0), \quad \mathbf{u}_1(x;1) = (0, 0, 0),$$

and

$$\mathbf{u}_0(x;2) = (x_1, x_2, x_3), \quad \mathbf{u}_1(x;2) = (0, x_2, x_3),$$

provided $(\rho, \lambda, \mu) \in \mathcal{E}$.

Indeed, since the initial data are harmonic functions, the terms with $\Delta$ in $\mathbf{D}$ are zero. Then using the condition $\lambda + \mu > \varepsilon_0$ one can drop this factor in the first column of $\mathbf{D}$. The needed $7 \times 7$-minor is now formed by the rows 2, 7, 8, 9, 10, 11, 12.

Conditions of Theorem 8.2.5 can relaxed in several ways. In particular, a local version (when the data are given on some part of $\partial\Omega$, like in Theorem 8.4.1) is available. Also instead of full Cauchy data one can prescribe on some part of $\partial\Omega$ zero Dirichlet boundary condition, etc. The Hölder stability estimate of Theorem 8.2.5 most likely can be improved to a Lipschitz one, which is quite promising for good numerics.

Local uniqueness and Lipschitz stability results are available for the more challenging problem with zero initial data; see the paper of Bukhgeim and Lavrent'ev [LaB], where the solution of the hyperbolic problem (8.0.1)–(8.0.3) (with $a_0 = a = 1, b = 0$) and its Neumann data are prescribed on the whole lateral boundary $\partial\Omega \times (0, T)$ when $T$ is large. These results are obtained by using integral geometry over the family of ellipsoids generated by intersections of backward and forward characteristic cones for the wave equation. The problems of integral geometry (with the data over curves joining points of the whole boundary) are stable (Theorem 7.1.4) as well as the Cauchy problem for nontrapping hyperbolic equations with the data on a "large" part of $\partial\Omega$ (Theorem 3.4.5). By using reduction to an inverse source problem and division and differentiation with respect to $t$ as in the proof of Theorem 8.2.2, one can hope to obtain good stability when determining coefficients of a hyperbolic equation. Yamamoto [Y] obtained certain "local" or "generic" results in this direction.

## 8.3 Many measurements: use of beam solutions.

We consider the operator

$$Au = -\Delta u + b_0\partial_t u + b \cdot \nabla u + cu.$$

Let $\mathcal{P}$ be a half-space in $\mathbb{R}^n$ and $\Omega_0 = \Omega \cap \mathcal{P}, \gamma_0 = \partial\Omega \cap \mathcal{P}$. We define the local lateral Neumann-to-Dirichlet map $\Lambda_\varepsilon(b, c; T) : g_1 \to g_)$ on $\gamma_0 \times (0, T)$, where $u$ is a solution to the mixed problem (8.0.1), (8.0.2), (8.0.3) with $f = 0$, zero initial data, and $\operatorname{supp} g_1 \subset \Gamma_0 \times (0, \varepsilon)$. In this section we assume that $b_0, b, c$ do not depend on $t$. By $\Lambda_{j,\varepsilon}$ we denote $\Lambda(b_j, c_j; T)$.

**Theorem 8.3.1.** *Suppose that $\Omega_0$ is simply connected, $b_0, b \in C^2(\overline{\Omega})$, $c \in L_\infty(\Omega)$,*

$$\operatorname{diam} \Omega_0 < T, \tag{8.3.1}$$

*and $b$ is given on $\gamma_0$.*

*Then the local Neumann-to-Dirichlet map $\Lambda_\varepsilon(b, c; T)$ uniquely determines $b_0$, curl $b$, $4c + b \cdot b - 2 \operatorname{div} b$ in $\Omega_0$.*

The proof is based on the following results.

**Lemma 8.3.2.** *$\Lambda_\varepsilon$ uniquely determines the integral*

$$\int_Q (-u(b_0 + b_0^*)\partial_t v - u(b + b^*) \cdot \nabla v + (c - \operatorname{div} b)uv) \tag{8.3.2}$$

*for any solution $u$ to the problem* (8.0.1)–(8.0.3) *with given $\partial_\nu u$ on $\partial\Omega \times (0, T)$,*

$$\partial_\nu u = 0 \text{ on } (\partial\Omega \backslash \gamma_0) \times (0, T) \text{ and in } C_0^2(\partial\Omega \times (0, T)), \tag{8.3.3}$$

*and any (given) solution $v \in H_{(2)}(Q)$ to the backward initial value problem for the hyperbolic equation*

$$(\Box + b_0^* \partial_t + b^* \cdot \nabla)v = 0 \text{ in } Q, \tag{8.3.4}$$

$$v = \partial_t v = 0 \text{ on } \Omega \times \{T\}, \tag{8.3.5}$$

$$\partial_\nu v = 0 \text{ on } (\partial\Omega \backslash \gamma_0) \times (0, T), \tag{8.3.6}$$

*where $\Box = \partial_t^2 - \Delta$ and $b_0^*, b^*$ are some given functions with the same properties as $b_0, b$.*

Proof. First observe that $\Lambda_\varepsilon$ uniquely determines $\Lambda_T$. Indeed, let us pick any $g_1$ supported in $\gamma_0 \times (0, T)$. Let represent it as the sum $g_{1,1} + \cdots + g_{1,m}$ with supp $g_{1,k}$ in $\gamma_0 \times [t_k, t_k + \varepsilon]$. Denote by $u_k$ a solution to the mixed hyperbolic problem (8.0.1)–(8.0.3) with large $T$ and with Neumann data $g_{1,k}$. Since the coefficients of the differential equation do not depend on $t$, the function $u_k(x, t - t_k)$ solves the mixed hyperbolic problem with the translated data $g_{1,k}(x, t - t_k)$ that are supported in $[0, \varepsilon]$. Then $\Lambda_\varepsilon$ uniquely determines $u_k(x, t - t_k)$ when $x \in \gamma_0$, $0 < t < T$, and therefore $u_k$ is uniquely determined on $\gamma_0 \times (0, T)$. Thus we obtain $u$ on $\gamma_0 \times (0, T)$. So we are given $\Lambda_T$.

Multiplying the equation for $u$ by $v$ and integrating by parts gives

$$\begin{aligned} 0 &= \int_Q ((\Box + b_0\partial_t + b \cdot \nabla + c)u)v \\ &= \int_{\partial\Omega \times (0,T)} (\partial_\nu vu - \partial_\nu uv + b \cdot \nu uv) \\ &\quad + \int_Q ((\Box - b_0\partial_t - b \cdot \nabla - \operatorname{div} b + c)v)u, \end{aligned}$$

where we used that $b_0$ does not depend on $t$ and the Cauchy data for $u$ are zero at $t = 0$ and for $v$ at $t = T$, so the integrals over $\Omega \times \{0\}$ and over $\Omega \times \{T\}$ are zero.

From conditions (8.3.3), (8.3.6), the integral over $\partial\Omega \times (0, T)$ is reduced to the integral over $\gamma_0 \times (0, T)$, where $\partial_\nu u$, $v$, $\partial_\nu v$ are given. The map $\Lambda_T$ then uniquely determines $u$ on this over part of the lateral boundary, and hence we are given the boundary integral. Using the equation for $v$, we complete the proof. □

**Lemma 8.3.3.** *For any $x \in \mathbb{R}^n$, any direction $\sigma$, and any function $\phi \in C_0^\infty(\mathbb{R}^n)$ there is a solution $u$ to equation* (8.0.1) *of the form*

$$u(x,t) = \phi(x + t\sigma)B(x,t)\exp(i\tau(x \cdot \sigma + t)) + r(x,t),$$

$$B(x,t) = \exp(-1/2 \int_0^t (b_0 + b \cdot \sigma)(x + s\omega)ds) \tag{8.3.7}$$

*with*

$$r = \partial_t r = 0 \text{ on } \Omega \times \{0\}, \qquad \partial_\nu r = 0 \text{ on } \partial\Omega \times (0, T), \tag{8.3.8}$$

*and*

$$\tau\|r\|_2(Q) + \|r\|_{(1)}(Q) \le C(\phi). \tag{8.3.9}$$

PROOF. Since

$$\Box(vw) = \Box v w + 2\partial_t v \partial_t w - 2\nabla v \cdot \nabla w + w\Box v$$

and $(\exp(i\tau(x \cdot \omega + t)) = 0$, equation (8.0.1) for $u$ is equivalent to the following equation for $r$:

$$(\Box + b_0\partial_t + b \cdot \nabla + c)r = F, \tag{8.3.10}$$

where

$$F = (-\Box - b_0\partial_t - b \cdot \nabla - c)(\Phi\exp(i\tau(x \cdot \sigma + t))$$
$$= -((i\tau(2\partial_t\Phi - 2\nabla\Phi \cdot \sigma + (b_0 + b \cdot \sigma)\Phi)$$
$$+ (\Box + b_0\partial_t + b \cdot \nabla + c)\Phi)\exp(i\tau(x \cdot \sigma + t)),$$
$$\Phi = \phi(x + t\sigma)B.$$

If $B$ is given by formula (8.3.7), then the factor of $i\tau$ in the above formula is zero. This is easy to derive by using the substitution $s = \theta + t$ in the integral defining $B$. So

$$F = F_1\exp(i\tau(x \cdot \sigma + t)), \qquad \|F_1\|_2(Q) + \|\partial_t F_1\|_2(Q) \le C. \tag{8.3.11}$$

Since the coefficients $b_0, b, c$ do not depend on $t$, the function

$$R(x,t) = \int_0^t r(x,s)ds$$

solves the mixed hyperbolic problem (8.3.10), (8.3.8) with the right side

$$F_2 = \int_0^t F = \int_0^t F_1(,s)(i\tau)^{-1}\partial_s(\exp(i\tau(x \cdot \sigma + s))ds.$$

Integrating by parts with respect to $s$ and using the bounds (8.3.11), we conclude that $\|F_2\|_2(Q) \leq C/\tau$. The standard energy estimates for mixed hyperbolic problems given by Theorem 8.1 imply then that $\|\partial_t R\|_2(Q) \leq C/\tau$, so we have the bound (8.3.9) for $r = \partial_t R$.

Since $\|F\|_2(Q) \leq C$, applying again the energy estimates we obtain the bound (8.3.9) for $\|r\|_{(1)}$ and thereby complete the proof. □

**Lemma 8.3.4.** *Assume that $b_0^*$, $b^*$ do not depend on $t$ and have the same regularity properties as $b_0$, $b$. For $x$, $\sigma$, $\phi$ in Lemma 8.3.3 there is a solution $v$ to the hyperbolic equation $(\Box + b_0^*\partial_t + b^* \cdot \nabla)v = 0$ in $Q$ of the form*

$$v(x,t) = \phi(x + t\sigma)B^*(x,t)\exp(-i\tau(x\cdot\sigma + t)) + r^*(x,t),$$

$$B^*(x,t) = \exp(-1/2\int_0^t (b_0^* + b^*\cdot\sigma)(x + s\sigma)ds) \tag{8.3.7*}$$

*with*

$$r^* = \partial_t r^* = 0 \textit{ on } \Omega\times\{T\}, \quad \partial_\nu r^* = 0 \textit{ on } \partial\Omega\times(0,T), \tag{8.3.8*}$$

*and*

$$\tau\|r^*\|_2(Q) + \|r^*\|_{(1)}(Q) \leq C(\phi). \tag{8.3.9*}$$

The proof is similar to that of Lemma 8.3.3.

Proof of Theorem 8.3.1.

Let $L$ be any straight line such that it intersection $L_0$ with $\overline{\Omega}_0$ is contained in the half-space $\mathcal{P}$. Let $\sigma$ be a direction of this line. We will show that our data determine the integral of $b_0 + b\cdot\sigma$ over $L$.

From our assumptions it is possible to find an interval $[y,z]$ in $\mathbb{R}^n$ containing $L_0$ such that $|y-z| < T$, and both $y$ and $z$ do not belong to $\overline{\Omega}_0$. Let us choose $\delta > 0$ so small that the $\delta$-neighborhoods of $y$ and $z$ do not intersect $\Omega_0$ as well, and the $\delta$-neighborhood of $[y,z]$ is contained in $\mathcal{P}$. Let $\phi \in C_0^\infty$ in the $\delta$-neighborhood of $y$ and zero elsewhere. We choose the direction $\sigma = -|z-y|^{-1}(z-y)$. Then the function $\phi(x+t\sigma)$ is zero in $(\mathbb{R}^n\backslash\overline{\mathcal{P}}\times(0,T)$ near $\overline{\Omega}\times[0,T]$. The boundary conditions (8.3.8) for $r$ guarantee that $u$ constructed in Lemma 8.3.3 satisfies all the conditions of Lemma 8.3.2. So does the function $v$, where we take $b_0^* = 0$, $b^* = 0$. According to Lemma 8.3.2, we are given the integrals (8.3.2). Using the formulae (8.3.7), (8.3.7*) and the bounds (8.3.9), (8.3.9*), we conclude that we are given

$$i\tau\int_Q (b_0 + b\cdot\sigma)(x)\phi^2(x+t\sigma)B(x,t)dxdt + \cdots, \tag{8.3.12}$$

where ... denotes the terms bounded with respect to $\tau$.

Dividing by $\tau$ and letting $\tau \to +\infty$ yields the integral (8.3.12). Extending $b_0$, $b$ as zero onto $\mathbb{R}^n\backslash\Omega_0$ and substituting $X = x + t\sigma, \theta = t$ in the integrals, we obtain the integral

$$\int \phi^2(X) \left( \int_0^T (b_0 + b \cdot \sigma)(X - \theta\sigma) B(X - \theta\sigma, \theta) d\theta \right) dX.$$

Since $\phi$ is an arbitrary smooth function supported near $y$, we are given the interior integral when $X = y$, i.e.,

$$\int_0^T (b_0 + b \cdot \sigma)(y - \theta\sigma) B(y - \theta\sigma, \theta) d\theta. \tag{8.3.13}$$

Using the substitution $s = \theta + s_1$ in the integral (8.3.7) defining $B$ and differentiating with respect to $\theta$, we obtain

$$\begin{aligned} d/d\theta\, B(y - \theta\sigma, \theta) &= d/d\theta \exp\left( -\frac{1}{2} \int_{-\theta}^0 (b_0 + b \cdot \sigma)(y + s_1\omega)\, ds_1 \right) \\ &= -\frac{1}{2}(b_0 + b \cdot \omega)(y - \theta\sigma) B(y - \theta\sigma, \theta). \end{aligned}$$

Therefore, the integral (8.3.13) is the difference of the value of the function $-2B(y - \theta\sigma, \theta)$ at the points $\theta = T$ and $\theta = 0$. The value at 0 is $-2$, so we are given the value at $\theta = T$. Taking the logarithm, we obtain

$$\int_L (b_0 + b \cdot \sigma).$$

Since the direction of $L$ is $-\sigma$ as well, we are given the integral of $b_0 - b \cdot \sigma$, and therefore the integrals of $b_0$ and of $b \cdot \sigma$ over all such $L$.

The next step is to show that these integrals determine $b_0$ and $\operatorname{curl} b$ in $\Omega_0$. First, we reduce the $n$-dimensional case to the two-dimensional one by intersecting $\Omega$ with two-dimensional planes and considering only lines $L$ in these planes. By Corollary 7.1.2 (where $K$ is the closure of the convex hull of $\Omega \backslash \mathcal{P}$) the integrals of $b_0$ over $L$ not crossing $\Omega \backslash \mathcal{P}$ uniquely determine $b_0$ in $\Omega_0$.

The recovery of $\operatorname{curl} b$ is based on the equality

$$\int_{p(\sigma)} \operatorname{curl} b = \int_L b \cdot \sigma,$$

where $p(\sigma)$ is a half-plane with exterior normal orthogonal to $\sigma$ that does not intersect $\Omega \backslash \mathcal{P}$. To prove this equality, introduce orthonormal coordinates $x_1, x_2$ such that the $x_1$-direction coincides with the exterior unit normal to $p(\sigma)$. Then the $x_2$-direction is parallel to $\sigma$. We have $\operatorname{curl} b = \partial_1 b_2 - \partial_2 b_1$. Integrating by parts, we obtain

$$\int_{p(\sigma)} \operatorname{curl} b = \int_L b_2 + \int_{p(\sigma) \cap \partial\Omega} (\nu_1 b_2 - \nu_2 b_1) = \int_L b \cdot \sigma + I.$$

Since the direction $(-\nu_2, \nu_1)$ is tangential to $\partial\Omega$, the integral $I$ is given by the conditions of Theorem 8.3.2. Given the integrals of $\operatorname{curl} b$ over all such $p(\omega)$, we can find the integrals of $\operatorname{curl} b$ over $L$ by considering parallel translations $p(\sigma)$ by

$\theta$ and then differentiating the integrals over the translated $p(\sigma)$ with respect to $\theta$. As above, the integrals over $L$ uniquely determine $\operatorname{curl} b$ in $\Omega_0$.

As in the proof of Theorem 5.4.1, $\operatorname{curl} b$ uniquely identifies $b_\bullet = b - \nabla\phi$, where $\phi$ is a $C^1(\overline{\Omega_0})$-function. We can assume that this function is zero at a point of $\Gamma_0$. Since $b$ is given there by the conditions of Theorem 8.3.1, both $\phi$ and $\partial_\nu\phi$ are given there as well. As in the proof of Theorem 5.5.1, the substitution $u = e^{\phi/2}v$ transforms the equation for $u$ into the equation

$$v + b_0\partial_t v + b_\bullet \cdot \nabla v + \left(c + \frac{1}{2}\operatorname{div} b_\bullet - \frac{1}{4}|b_\bullet|^2 - \frac{1}{2}\operatorname{div} b + \frac{1}{4}|b|^2\right)v = 0.$$

Since $\phi$ and $\partial_\nu\phi$ are given on $\gamma_0$, the Neumann-to-Dirichlet map for the original equation uniquely determines the map for the new equation. To complete the proof, it suffices to show that $c - \frac{1}{2}\operatorname{div} b + \frac{1}{4}|b|^2$ is uniquely determined when $b_0, b_\bullet$ are given. We will again make use of integrals (8.3.2), where now $b_0, b = b_\bullet$ are given. We will use in the integrals (8.3.2) new functions $u_\bullet$ given in Lemma 8.3.3 for the $\bullet$-equation and solutions $v$ to the hyperbolic equation (8.3.4) with $b_0^* = -b_0$, $b^* = -b$. Multiplying in the integrand, we will have

$$\phi^2(x + t\omega)(c(x) + \operatorname{div} b_\bullet - \frac{1}{4}|b_\bullet|^2 - \frac{1}{2}\operatorname{div} b + \frac{1}{4}|b|^2) + O(\tau^{-1}).$$

Therefore, we are given integrals of the first term over $Q$. Repeating the argument from the beginning of the proof of Theorem 8.3.1, we conclude that we know (nonweighted) integrals of $c$ over all straight lines $L$ joining boundary points in $\gamma_0$. So, as above, uniqueness in the Radon transform guarantees that $c - \frac{1}{2}\operatorname{div} b + \frac{1}{4}|b|^2$ is unique in $\Omega_0$.

The proof is complete. □

Observe that using $L_2(Q)$-bounds of solutions of hyperbolic initial boundary value problems with source terms in negative Sobolev spaces $H_{(-1)}(Q)$ (see the book of Lions and Magenes [LiM], Vol. 1, Theorem 9.3, p. 288) one can obtain uniqueness of $b_0, b \in C^1(\overline{\Omega_0})$. One has only to make minor changes in the proofs of Lemmas 8.3.3, 8.3.4. Using the structure of fundamental solutions of hyperbolic equations and integral geometry, Romanov [Rom] obtained uniqueness of $\operatorname{curl} b$ and $c - \frac{1}{2}\operatorname{div} b + \frac{1}{4}|b|^2$ in 1974. Then Rakesh and Symes showed uniqueness of $c(b = 0, b^0 = 0)$ in the paper [RS], where they first used beam solutions in inverse hyperbolic problems. In these papers $\Omega_0 = \Omega$; i.e., they consider the data on the whole lateral boundary. Beam solutions seem to be a very useful tool in inverse problems that has not yet been completely utilized. For review of the subject we can refer to the books of Arnaud [Ar], Katchalov, Kurylev, and Lassas [KKL] and to the paper of Ralston [Ra]. The idea to use rapidly oscillating solutions in the inverse hyperbolic problems with given results of all boundary measurements originated from the paper of Sylvester and Uhlmann [SyU2] on elliptic equations, which has been discussed in Sections 5.2, 5.3.

To formulate a stability estimate, we impose the following constraints on $b_{0j}, c_j$ : $|b_{0j}|_2(\Omega) + |c_j|_1(\Omega) < M$, $j = 1, 2$.

We view $\Lambda_{j,\varepsilon}$ as an operator from $L_2(\Gamma_0 \times (0, \varepsilon))$ into $H_{(1/6)}(\Gamma_0 \times (0, T))$ and denote by $\delta$ the operator norm of $\Lambda_{2,\varepsilon} - \Lambda_{1,\varepsilon}$.

**Theorem 8.3.5.** *There are $C, \lambda$ depending only on $P, \Omega, \varepsilon, M$ such that*
*(a) When $n = 3$, we have $|b_{02} - b_{01}|_0(\Omega_0) + |c_2 - c_1|_0(\Omega_0) \leq C\delta^{\lambda}$.*
*(b) When $n = 2$ and $b_{01} = b_{02}$, we have $|c_2 - c_1|_0(\Omega_0) \leq C(-\ln \delta)^{-\lambda}$.*

This result is proven in the paper of Isakov and Sun [IsS1] using the scheme of the proof of Theorem 8.3.1. Observe that the (Hölder) stability estimates in the three-dimensional case are much better than in the two-dimensional one and in the inverse conductivity problem (Theorem 5.2.3). We think that by modifying the method of the proof one can obtain Hölder-type stability in the two-dimensional case as well.

When one is given the complete Dirichlet-to-Neumann map for any initial data and the lateral Neumann data, and one measures the Dirichlet data and the Cauchy data at the final moment of time, then there are results about uniqueness of the coefficient $c(x, t) \in L_\infty(Q)$ (see, e.g., [Is7]) that are similar to Lemma 9.6.3.

The beam solutions can be constructed for more general hyperbolic equations. In principle, this construction shows that for scalar $a$ the lateral Neumann-to-Dirichlet map (when $T$ is sufficiently large) uniquely determines geodesic distances between point of $\gamma_0$, which is not enough to obtain uniqueness of $a$ in $\Omega_0$ without additional assumptions.

When $\gamma_0 = \partial\Omega$, there is another way to show that $\Lambda_l$ (when $T$ is large) uniquely determines geodesic distances between points of $\partial\Omega$. A simple argument based on the structure of the fundamental solution of the Cauchy problem for second-order hyperbolic equations (progressive wave expansion) is given by Uhlmann [U1]. We will give a minor part of it as the following exercise.

**Exercise 8.3.6.** Consider the hyperbolic equation (8.0.1) with the given lateral Neumann-to-Dirichlet map. Show that a solution of the Cauchy problem for this equation in $\mathbb{R}^n \times (0, T)$ with the (given) data $u_0, u_1, f$ supported in $\mathbb{R}^n \backslash \overline{\Omega}$ is uniquely determined in $(\mathbb{R}^n \backslash \Omega) \times (0, T)$.

Given all geodesic distances on $\partial\Omega$, a (conformal, or scalar) Riemannian metric $a(x)|dx|^2$ is uniquely determined, provided that its geodesics are regular (see, e.g., [Rom, p. 93]). So if $\Gamma_0 = \partial\Omega$, $T$ is large, and $a$ generates a regular geodesic, the lateral Neumann-to-Dirichlet map uniquely determines $a$. Recently, Eskin [Es2] showed that one also can recover unknown inclusions and established a connection with Aharonov-Bohm effect from theoretical physics.

For a discussion of the uniqueness problem for a matrix $a$ we refer to the paper of Sylvester and Uhlmann [SyU4].

## 8.4 Many measurements: methods of boundary control

We consider the initial value problem (8.0.1)–(8.0.3), where
(8.4.1)

$$Au = -a\Delta, a_0 = 1, a \in C^\infty(\overline{\Omega}) \text{ does not depend on } t, \text{ and } a > 0 \text{ on } \overline{\Omega}.$$

We assume zero initial conditions $u_0 = 0$, $u_1 = 0$, which is natural in many applications (say, in geophysics). We define the Riemannian distance

$$d_a(x, y) = \inf \int_{\gamma(x,y)} a^{-1/2} d\gamma$$

where inf is over all smooth regular curves $\gamma$ in $\Omega$ with endpoints $x, y$. Let $\gamma_1 = \partial\Omega$ and $\Lambda^{-1}_{\gamma_0,T}$ be the local lateral Neumann-to-Dirichlet map corresponding to a part $\gamma_0 \subset \partial\Omega$ : supp $g_1 \subset \gamma_0 \times (0, T)$ and $g_0$ is given only on $\gamma_0 \times (0, T)$.

**Theorem 8.4.1.** *The lateral Neumann-to-Dirichlet map $\Lambda^{-1}_{\gamma_0,l}$ uniquely determines a in the domain $\Omega_{\gamma_0,T} = \{y \in \Omega : d_a(y, \gamma_0) < T/2\}$.*

This result belongs to Belishev [Be1], [Be2], [Be3]. We will reproduce a modified scheme of his proof. It is more convenient to assume that we are given the local lateral Dirichlet-to-Neumann map $\Lambda_{\gamma_0,T} : g_0 \to g_1$.

*Outline of proof.* Let us assume that

$$\Omega_T \neq \Omega. \tag{8.4.2}$$

This is certainly true for small $T$. By reconstructing $a$ for small $T$ and then increasing $T$, we will cover the general case. For small $T$ one can introduce in $\Omega_T$ the geodesic coordinates $x(\eta, \xi)$, $\eta \in \partial\Omega$, $\xi = \text{dist}_a(x, \partial\Omega)$.

Let $W_{\gamma_0,T} g_0 = u(, T; g_0)$ where $u(; g_0)$ be the solution to the hyperbolic problem (8.0.1), (8.0.2), $u = g_0$ on $\partial\Omega \times (0, T)$, supp $g_0 \subset \gamma_0 \times (0, T)$ on $\Omega \times \{T\}$. We introduce the weighted space $L_{2,a}(\Omega)$ by using the scalar product

$$(u, v)_{2,a}(\Omega) = \int_\Omega a^{-1} uv$$

in $L_2(\Omega)$. In fact, it is the same space $L_2(\Omega)$ with a different scalar product and (equivalent) norm. As observed in section 8.0, the operator $W_{\gamma_0,T}$ is linear and continuous from $L_2(\partial\Omega \times (0, T))$ into $L_{2,a}(\Omega)$, so we can define the operator $C_{\gamma_0,T} = W^*_{\gamma_0,T} W_{\gamma_0,T}$ ( where $W^*$ is the adjoint of $W$) from $L_2(\gamma_0 \times (0, T)$ into itself. We have

$$(C_{\gamma_0,T} g_{01}, g_{02})_2(\partial\Omega \times (0, T)) = (W_{\gamma_0,T} g_{01}, W_{\gamma_0,T} g_{02})_{2,a}(\Omega). \tag{8.4.3}$$

By $C_{g_0}$ we denote the restriction of $C$ onto $g_0$ supported in $\gamma_0$.

**Lemma 8.4.2.** *The operator $C_{\gamma_0,T}$ is a symmetric, positive linear operator in $L_2(\gamma_0 \times (0, T))$. It has zero kernel and is uniquely determined by $\Lambda_{\gamma_0,2T}$.*

PROOF. Assume that $Cg_0 = 0$. Then $(Wg_0, Wg_0)_{2,a}(\Omega) = 0$ and $Wg_0 = 0$. We will extend the solution $u = u(; g_0)$ onto $\Omega \times \mathbb{R}$ as follows. Let us define $u(x, t) = -u(x, 2T - t)$ when $T \le t < 2T$, and $u = 0$ when $2T \le t$. Then the limits of $u(x, t)$ as $t \to T$ from $t < T$ and $T < t$ are 0, and the limits of $\partial_t u(x, t)$ are equal. By using the definition of a generalized solution, we conclude that $u$ solves the same hyperbolic equation in $\Omega \times \mathbb{R}$. By assumption (8.4.2), the set $\Omega \backslash \Omega_T$ is not empty. Apparently, $u = 0$ on this set for all $t \in \mathbb{R}$. The Laplace transform $U(x, s)$ of $u$ with respect to $t$ satisfies the elliptic equation $s^2 U - a\Delta U = 0$ in $\Omega \times \mathbb{R}$, and it is zero on $\Omega \backslash \Omega_T$. By uniqueness of the continuation for elliptic equations (Section 3.3) we have $U = 0$ on $\Omega$. By inverting the Laplace transform we obtain $u = 0$, and so its Dirichlet data $g_0$ are zero as well.

To show that $\Lambda_{\gamma_0,2T}$ uniquely determines $C_{\gamma_0,T}$, observe that

$$
\begin{aligned}
&(\partial_t^2 - \partial_s^2)(u(, t; g_{01}), u(, s; g_{02}))_{2,a}(\Omega) \\
&= \int_\Omega ((\Delta u(, t; g_{01}))u(, s; g_{02}) - u(, t; g_{01})\Delta u(, s; g_{02})) \\
&= \int_{\gamma_0} ((\partial_\nu u(, t; g_{01}))u(s; g_{02})) - u(, t; g_{01})\partial_\nu u(, ; g_{02})) \\
&= \int_{\gamma_0} (g_{01}(t)\Lambda_{\gamma_0,2T} T g_{02}(s) - \Lambda_{\gamma_0,2T} g_{01}(t) g_{02}(s))
\end{aligned}
$$

is given for $s < 2T, t < 2T$ when we are given $g_{01}, g_{02}$. In addition, $u(t; g_{01}) = 0$ when $t < 0$. By solving the Cauchy problem for the one-dimensional wave operator $(\partial_t^2 - \partial_s^2)$ we uniquely determine $(u(, t; g_{01}), u(, s; g_{02}))_{2,a}(\Omega)$ when $t < 2T, s + t < 2T$. Letting $s = y = T$, we obtain $(W_{\gamma_0,T} g_{01}, W_{\gamma_0,T} g_{02})_{2,a}(\Omega)$, which uniquely determine $C_{\gamma_0,T}$ by (8.4.3).

The proof is complete. □

**Exercise 8.4.3.** Prove that for any harmonic function $v$ in $\Omega$, $v \in C^1(\overline{\Omega})$, $v = 0$ on $\partial\Omega \backslash \gamma_0$ we have

$$(u(, T; g), v)_{2,a}(\Omega) = (g, \Lambda^\bullet v)_2(\Gamma_0 \times (0, T))$$

where $\Lambda^\bullet v = \Lambda^{-1*}\theta v - \theta \partial_\nu v$, $\theta(t) = (T - t)$.

{*Hint*: integrate by parts in

$$0 = \int_{\Omega \times (0,T)} (a^{-1}\partial_t^2 u - \Delta u)\theta v\}$$

This result is important for the proof of Theorem 8.4.1 because it enables to “replace” unknown function $u(, T; g_0)$ by known function $v$.

Let $\gamma$ be an open smooth subsurface of $\gamma_0$ and $W_{\gamma,T}$ the operator $W$ considered only on functions $g_0$ that are zero outside $\gamma \times (0, T)$.

**Lemma 8.4.4.** *The range of $W_{\gamma,T}$ is dense in $L_{2,a}(\Omega_{\gamma,2T})$.*

PROOF. Let us assume the opposite. Then by the Hahn-Banach theorem, there is a function $u_1 \in L_2(\Omega)$ that is $L_{2,a}(\Omega)$-orthogonal to all solutions $u(, T; g_0)$ with $g_0 = 0$ outside $\gamma \times (0, T)$. Since all these solutions are zero outside $\Omega_{\gamma,2T}$, we can let $u_1$ be zero onto $\Omega \backslash \Omega_{\gamma,2T}$.

Let us consider the solution $v$ to the backward initial boundary value problem

$$
\begin{aligned}
(\partial_t^2 - a\Delta)v &= 0 \text{ on } Q,\\
v &= 0,\ \partial_t v = u_1 \text{ on } \Omega \times \{T\},\\
v &= 0 \text{ on } \partial\Omega \times (0, T).
\end{aligned}
\tag{8.4.4}
$$

When $u_1 \in \mathring{H}_{(1)}(\Omega)$ and $g_0 \in \mathring{H}_{(1)}(\Gamma \times (0, T))$, we have $v, u \in H_{(2)}(Q)$. Multiplying the equation $\partial_t^2 - a\Delta v = 0$ by $a^{-1}u$ and using integration by parts with respect to $t$ and Green's formula with respect to $x$, we obtain

$$
\begin{aligned}
0 &= \int_Q a^{-1}(\partial_t^2 - a\Delta)vu\\
&= \int_{\Omega\times\{T\}} a^{-1}u_1 u - \int_{\partial\Omega\times(0,T)} (\partial_\nu v u - \partial_\nu u v)\\
&= \int_{\Omega\times\{T\}} a^{-1}u_1 u - \int_{\gamma\times(0,T)} g_0 \partial_\nu v.
\end{aligned}
\tag{8.4.5}
$$

Some integrals over the boundary are zero due to the zero initial and boundary value conditions (8.0.2), (8.4.4) for $u$ and $v$. So for all such regular solutions the last expression is zero. Approximating $u_1$ by $\mathring{H}_{(1)}$-functions in $L_2(\Omega)$, we can obtain this equality in the less regular case under consideration. By the choice of $u_1$, we have

$$
\int_{\gamma\times(0,T)} g_0 \partial_\nu v = \int_{\Omega\times\{T\}} a^{-1}u_1 u = 0
$$

when $u = u(, T; g_0)$, $g_0 \in \mathring{H}_{(1)}(\partial\Omega \times (0, T))$, $g_0 = 0$ outside $\gamma \times (0, T)$. Since the space of such $g$ is dense in $L_2(\gamma \times (0, T))$, we conclude that $v = 0$ on $\gamma \times (0, T)$. Therefore, $v$ has zero Cauchy data on $\gamma \times (0, T)$. As above, we extend $v$ onto $\Omega \times (T, 2T)$ by letting $v(x, t) = -v(x, 2T - t)$. Then using mollifying (described in the book of Hörmander [Hö2]) with respect to $t$ and Theorem 8.1, we can assume that $v \in H_{(2)}(\Omega \times (0, T))$.

Now we can apply Corollary 3.4.6 to conclude that $v = 0$, $\partial_t v = 0$ on $\Omega_{\gamma,2T} \times \{T\}$, because any point of this surface can be reached by noncharacteristic deformations of $\gamma \times (0, 2T)$. So $u_1 = \partial_t v = 0$, which contradicts the choice of $u_1$.

The proof is complete. □

In the proof we will use the following known (see, e.g. [Be3]) formula for propagation of jump singularities of solutions $v$ to the problem (8.4.4)

$$
\lim_{\tau\to T-\xi} \partial_\nu v(\eta, \tau)(P_T - P_\xi)\phi = \alpha(\eta, \xi)\phi(x(\eta, \xi), T),\ \eta \in \gamma_0
\tag{8.4.6}
$$

provided $\phi \in C^\infty(\overline{\Omega})$, $0 < \xi < T$, and $\alpha$ is a positive function determined only by $a$. Here $P_\xi$ is the projector defined as the operator of multiplication by the function $\chi(\Omega_{\gamma_0,2\chi})$. On the other hand, from (8.4.5) we have

$$(W_{\gamma_0,T}g_0, \phi)_{2,a}(\Omega) = (g_0, \partial_\nu v)_2(\gamma_0 \times (0,T))$$

for all $g_0 \in L_2(\gamma_0 \times (0,T))$, so $\partial_\nu v = W^*_{\gamma,T}\phi$. Hence we have the relation

$$\partial_\nu v(;(P_T - P_\xi)\phi) = W^*_{\gamma_0,T}(P_T - P_\xi)\phi \tag{8.4.7}$$

A crucial observation is that $\Lambda_{\gamma_0,2T}$ uniquely determines the right side of (8.4.7). Indeed, let $g_{0j}(;\xi)$ be a complete system of functions in $L_2(\gamma_0 \times (T-\xi, T))$ which is orthonormal in the following sense: $(C_{\gamma_0,2T}g_{0j}, g_{0k})_2(\gamma_0 \times (T-\xi,T)) = \delta_{jk}$ (Kronecker delta). Such a system can be obtained from any $L_2(\gamma_0 \times (T-\xi,T))$-complete system by using the Gram-Schmidt process. Lemma 8.4.2 guarantees that $(C\gamma_0, Tg01, g_{02})_2$ is a scalar product. Lemma 8.4.2 and Lemma 8.4.4 also imply that $W\gamma_0, Tg_{0j}(;\xi)$ form an orthonormal basis in $L_{2,a}(\Omega(\gamma_0,\chi))$. We have

$$\begin{aligned}
&W^*_{\gamma_0,T}(P_T - P_\xi)\phi\\
&= W^*_{\gamma_0,T}\Bigg(\sum_{j=1}^{\infty}(\phi, W_{\gamma_0,T}g_{0j}(;T))_{2,a}W_{\gamma_0,T}g_{0j}(;T)\\
&\quad -(\phi, W_{\gamma_0,T}g_{0j}(;\xi))_{2,a}W_{\gamma_0,T}g_{0j}(;\xi)\Bigg)\\
&= \sum_{j=1}^{\infty}((\phi, W_{\gamma_0,T}g_{0j}(;T))_{2,a}C_{\gamma_0,T}g_{0j}(;T)\\
&\quad -(\phi, W^T g_{0j}(;\xi))_{2,a}C_{\gamma_0,T}g_{0j}(;\xi))
\end{aligned}$$

Hence, due to Lemma 8.4.2 and Exercise 8.4.3, for any harmonic function $\phi \in C^\infty(\overline{\Omega})$, $\phi = 0$ on $\partial\Omega\backslash\gamma_0$, the operator $\Lambda_{\gamma_0,2T}$ uniquely determines $W^{T*}(P_T - P_\xi)\phi$. Combining this observation with the relations (8.4.6), (8.4.7) we conclude that $\Lambda_{\gamma_0,2T}$ uniquely determines $\alpha(\eta,\xi)\phi(x(\eta,\xi),T)$ for any such $\phi$.

We will complete the reconstruction in the normal neighborhood $\mathcal{N}(T)$ of $\gamma_0$ by using appropriate $\phi$. Here the normal neighborhood is defined as $\{x(\eta,\xi) : \eta \in \gamma_0, 0 < \xi < T/2\}$. Let $K$ be any compact in $\Omega \cup \gamma_0$. From the Runge property for harmonic functions (see the proof of Lemma 5.7.2) and interior Schauder estimates of Theorem 4.1 it follows that we can approximate the harmonic functions $1, x_1, \ldots, x_n$ in $C^1(K)$ by harmonic $C^\infty(\overline{\Omega})$-functions $\phi_0, \ldots, \phi_n$ which are zero on $\partial\Omega\backslash\gamma_0$. Therefore we can choose $\phi_0, \ldots, \phi_n$ in such a way that $\phi_0 > \frac{1}{2}$ and the map $x \to (\phi_1(x), \ldots, \phi_n(x))$ is one-to-one on $K$. We already found that $\Lambda_{\gamma_0,T}$ uniquely determines $\phi_j/\phi_0(x(\eta,\xi))$, $j = 1, \ldots, n$ when $\eta \in \gamma_0$, $\xi < T/2$. Due to the choice of $\phi_j$ it uniquely determines $x(\eta,\xi)$ for these $\eta, \xi$. Since the map $x(\eta,\xi)$ is one-to-one we can uniquely find $\xi = \xi(x)$, $x \in \mathcal{N}(T)$. Then by the eikonal equation

$$a^{-1}(x) = |\nabla_x \xi|^2, x \in \mathcal{N}(T).$$

To show uniqueness in the remaining part of $\Omega_{\gamma_0,T}$ and for larger $T$ we can use a smaller domain $\Omega_2 \subset \Omega$ provided $a$ is known in $\Omega\backslash\Omega_2$.

Let $\Omega_2$ be a subdomain of $\Omega$ such that $\gamma_2 = \partial\Omega_2\backslash\partial\Omega$ is in $\Omega$, $\partial\Omega_2 \in C^\infty$, and any geodesic $x(\eta, \xi)$, $\eta \in \gamma_0, 0 < \xi < T$, intersects $\Omega\backslash\Omega_2$ over a connected set. Let $Q_0$ be $\cup\Omega_{\gamma_0,T-|2t-T|} \times \{t\}$ over $t \in (0, T)$ and $\Gamma_{20} = \gamma_2 \times (0, T) \cap Q_0$. We claim that if $a$ is given on $\Omega\backslash\Omega_2$ then the Dirichlet-to-Neumann operator $\Lambda_2 : g_{02} \to \partial_\nu u_2$ on $\gamma_{20}$ is uniquely determined by the original Neumann-to-Dirichlet operator. Here $u_2$ is the solution to the following hyperbolic problem

$$\partial_t^2 u_2 - a\Delta u_2 = 0 \text{ in } \Omega_2 \times (0, T)$$
$$u_2 = \partial_t u_2 = 0 \text{ on } \Omega_2 \times \{0\}$$
$$u_2 = g_{02} \text{ on } \partial\Omega_2 \times (0, T), \operatorname{supp} g_{02} \subset \gamma_{20}$$

Indeed, let $u$ be the solution of the above problem with $\Omega_2, g_{02}, \gamma_{20}$ replaced by $\Omega, g_0, \gamma_0 \times (0, T)$. The original Dirichlet-to-Neumann map uniquely determines $\partial_\nu u$ on $\gamma_0 \times (0, T)$ when $g_0$ is given. Due to our assumptions about $\Omega_2$ the set $Q_0$ can be covered by differentiable family of noncharacteristic surfaces with the boundaries on $\gamma_0 \times (0, T)$. Since the hyperbolic equation is known in $\Omega\backslash\Omega_2 \times (0, T)$ Corollary 3.4.6 guarantees that $u$ is uniquely determined in $Q_0$. Hence for any $g_{02} = u$ on $\partial\Omega_2 \times (0, T)$ we are given $\partial_\nu u$ on $\gamma_{20}$. Now our claim follows from completeness of such $u$ in $L_2(\gamma_{20})$ guaranteed by Lemma 8.4.5.

**Lemma 8.4.5.** *Any function* $g_{02} \in L_2(\gamma_2 \times (0, T))$ *with* $\operatorname{supp} g_{02} \subset \Gamma_{20}$ *can be* $L_2$*-approximated on* $\Gamma_{20}$ *by functions u solving the equations* (8.4.1) *on* $\Omega \times (0, T)$ *with zero initial conditions on* $\Omega \times \{0\}$ *and with* $u = 0$ *on* $(\partial\Omega\backslash\gamma_0) \times (0, T)$.

PROOF. Let us assume the opposite. Then there is a non-zero function $\psi \in L_2(\gamma_2 \times (0, T))$, $\operatorname{supp}\psi \subset \Gamma_{20}$ such that

$$\int_{\gamma_2\times(0,T)} \psi u = 0 \tag{8.4.8}$$

for all $u$ described in Lemma 8.4.5. Using completeness of smooth functions in $L_2$ we can assume that (extended) $\psi \in C^\infty(\mathbb{R}^{n+1})$.

Let $u^*$ be the solution of the following transmission problem

$$\partial_t^2 u^- - a\Delta u^- = 0 \text{ in } \Omega_2^- \times (0, T), \partial_t^2 u^+ - a\Delta u^+ = 0 \text{ in } \Omega_2^+ \times (0, T)$$
$$u^- = \partial_t u^- = 0 \text{ on } \Omega_2^- \times \{T\}, u^+ = \partial_t u^+ = 0 \text{ on } \partial\Omega_2^+ \times \{T\}$$
$$u^- = u^+, \partial_\nu u^+ - \partial_\nu u^- = \psi \text{ on } \partial\Omega_2^+ \times (0, T)$$
$$u^- = 0 \text{ on } \partial\Omega \times (0, T). \tag{8.4.9}$$

Here $\Omega_2^+$ is a smooth subdomain of $\Omega_2$ with the closure in $\Omega$ such that $\operatorname{supp}\psi \subset \partial\Omega_2^+$, $\Omega_2^- = \Omega\backslash\overline{\Omega_2^+}$ and $u^* = u^-$ on $\Omega_2^- \times (0, T)$, $u^* = u^+$ on $\Omega_2^+ \times (0, T)$. By subtracting $\psi$ from $u_2^-$ and using Theorem 8.1 we obtain existence of a (weak) solution to the transmission problem with $\partial_t^k u^* \in L_2(\Omega_2 \times (0, T))$. By subtracting $(1 + a)\partial_t^2 u^-$, $(1 + a)\partial_t^2 u^+$ from the both sides of the differential equations (8.4.9) we can consider these equations and the relations on $\partial\Omega_2^+$ as an elliptic transmission problem. The solution of the elliptic transmission problem is the sum of volume potentials over $\Omega_2^+ \times (0, T)$, $\Omega_2^- \times (0, T)$ and of the single layer potential $U$ with density $\psi$ over $\partial\Omega_2^+ \times (0, T)$. The known properties of potentials (for the Laplace equation) imply that for the sequence of domains $\Omega_{2k}^+$ whose boundaries are images of $\partial\Omega_2^+$ under the maps $\Phi(x; k) = x - k^{-1}\nu(x)$ (interior normal deformation) we have $|\nabla U(\Phi(; k), )| < g \in L^2(\partial\Omega_2^+ \times (0, T))$ and the functions $\nabla U(\Phi(; k), )$ have limits almost everywhere on $\partial\Omega_2^+ \times (0, T)$ when $k \to \infty$. The similar property holds for outside limits. From theory of elliptic boundary value problems it follows that $u_2^+ \in H_{(2)}(\Omega_3^+ \times (0, T))$, $u_2^- \in H_{(2)}(\Omega_3^- \times (0, T))$ where $\Omega_3^+, \Omega_3^-$ are subdomains of $\Omega_2^+, \Omega_2^-$ such that their closures are inside $\Omega_2^+, \Omega_2^- \cup \partial\Omega$. From the differential equation (8.4.9) we have

$$0 = \int_{\Omega_3^+ \times (0,T)} (a^{-1}\partial_t^2 u^+ - \Delta u^+)u = \int_{\partial\Omega_3^+ \times (0,T)} (\partial_\nu u u^+ - u\partial_\nu u^+)$$

where we did integrate by parts two times, have used zero initial and final conditions (8.4.9) for $u$ and $u^+$ and the differential equation for $u$. Letting $\Omega_3^+ = \Omega_{2k}^+$ and passing to the limit as $k \to \infty$ we obtain that

$$\int_{\partial\Omega_2^+ \times (0,T)} (\partial_\nu u u^+ - u\partial_\nu u^+) = 0 \tag{8.4.10}$$

We can similarly consider $u^-$ to obtain

$$-\int_{\gamma_0 \times (0,T)} u\partial_\nu u^- - \int_{\partial\Omega_2^- \times (0,T)} (\partial_\nu u u^- - u\partial_\nu u^-) = 0 \tag{8.4.11}$$

where we have used that $\partial\Omega^- = \partial\Omega \cup \partial\Omega^+$, that $u^- = 0$ on $\partial\Omega$ and that $u = 0$ on $\partial\Omega\backslash\Gamma_0$. Adding the equalities (8.4.10) and (8.4.11) and using the transmission condition (8.4.9) we conclude that

$$-\int_{\gamma_0 \times (0,T)} u\partial_\nu u^- - \int_{\partial\Omega_2^+ \times (0,T)} \psi u = 0$$

From (8.4.8) we conclude that the first integral is zero. Since $u$ on $\gamma_0 \times (0, T)$ can be any smooth function we have $\partial_\nu u^- = 0$ on $\gamma_0 \times (0, T)$. Using also the boundary condition (8.4.9) we obtain that $u^-$ has zero Cauchy data on $\gamma_0 \times (0, T)$. As above from uniqueness in the lateral Cauchy problem we derive that $u^- = 0$ on $Q_0 \cap (\partial\Omega_2^+ \times (0, T))$.

Now we will complete the proof by showing that $\psi = 0$. Our first claim is that $u^- = 0$ on $\partial\Omega_2^+ \times (T/2, T)$. Indeed, we have this equality on $\Gamma_{20}$. Due to the definition $\Gamma_{20} \cap \{T/2 < t < T\}$ is the intersection of $\partial\Omega_2^+ \times (T/2, T)$ and of the influence domain for our hyperbolic equation. Since $\psi$ is supported in $\Gamma_{20}$ we conclude that a solution $u^-$ to the backward hyperbolic problem (8.4.9) is zero on $\partial\Omega_2^+ \times (T/2, T)\backslash\Gamma_{20}$. Finally, $u^+ = u^- = 0$ on $\partial\Omega_2^+ \times (T/2, T)$ and therefore a solution $u^+$ to the hyperbolic equation (8.4.9) with zero final conditions is zero on $\Omega_2^+ \times (T/2, T)$. Therefore from the transmission condition (8.4.9) we derive that $\psi = 0$ on $\partial\Omega_2^+ \times (T/2, T)$. To show that $\psi$ is zero when $0 < t < T/2$ we consider the hyperbolic equation (8.4.9) for $u_2^+$ in the intersection $Q_+$ of $\Omega_2^+ \times (0, T/2)$

with the influence domain of $\Gamma_{20} \cap \{0 < t < T/2\}$. We have zero Cauchy data on $\Omega_2 \times \{T/2\}$ and zero lateral boundary data $u^+ = u^- = 0$ (on $\Gamma_{20}$). By uniqueness in the mixed hyperbolic problem $u^+ = 0$ on $Q_+$. Now from the transmission conditions (8.4.9) and from our previous conclusions about $u_2^-$ we derive that $\psi = 0$ on the remaining part of $\Gamma_{20}$. □

Observe that the assumption $a \in C^\infty(\overline{\Omega})$ is needed only for the formula (8.4.6) (in fact $C^N$ with large $N$ is sufficient). All other parts of the proof require only $a \in C^1(\overline{\Omega})$.

By extending this method Belishev and Kurylev [Be3], [BeK], [KKL] obtained a complete solution of uniqueness problem for an anisotropic equation with complete spectral data or with the lateral hyperbolic Dirichlet-to-Neumann map. They proved the following result

**Theorem 8.4.6.** *Let $a$ be a $C^\infty(\overline{\Omega})$ positive symmetric matrix. Let $a^\bullet$ be another matrix of the coefficients of the operator $A$ which produces the same complete spectral data (eigenvalues and boundary values of normalized Neumann eigenfunction) or the same lateral Neumann-to-Dirichlet map as $a$. Then there is an isometry of the Riemannian manifold $(\Omega, a)$ onto the Riemannian manifold $(\Omega, a^\bullet)$ which is identical on $\partial\Omega$*

There are other of recovery of the speed of the propagation which are not using the formula (8.4.6). In particular, in [Be3] a so-called mark function (a harmonic function with singularity inside $\Omega$ and with zero Dirichlet data on $\partial\Omega$ is used). Also, Gaussian beams can be incorporated in the reconstruction process [KKL].

The anisotropic inverse hyperbolic problem has been considered by Sylvester and Uhlmann [SyU4], who obtained uniqueness results for $a$ in a linearized (around the Euclidean metric) version of the inverse problem. They showed that by using the progressive wave expansion for the Cauchy problem one can find Riemannian distances between all boundary points of $\Omega$. Then from known results [Rom] one concludes that a conformal metric ($a$ is scalar) is unique, and by using uniqueness in the Radon transform and a harmonic map, one can prove the linearized version in the general case.

## 8.5 Recovery of discontinuity of the speed of propagation

An important applied problem concerns recovery of a discontinuity surface of the speed of propagation from boundary observations.

We consider the hyperbolic equation (8.0.1), where

$$Au = -\operatorname{div}((1 + k\chi(D \times (-\infty, T)))\nabla u), \ a_0 = 1$$

in the half-space $\mathbb{R}^3_- \times (-\infty, T)$ with zero initial data $u = 0$ when $t < 0$. Given $R > 0$, we let $\gamma_0 = \partial\Omega \cap B(0; R)$, so the Neumann data $g_1$ are supported in $B(0; R) \times [0, T]$.

With respect to domains $D_j$, we assume that they are subgraphs $\{(x,t): x_3 < d_j(x_1,x_2)\}$ of Lipschitz functions $d_j$. We introduce the uniqueness set

$$U = \{-T/2 < x_3, |x'| < R\}.$$

We will make use of the special Neumann data. To describe them we consider any function $\phi \in H_{(2)}(\mathbb{R})$ such that it is zero on $(-\infty, 0)$ and $\phi, \phi'$ are positive on the interval $(0,\tau)$ for some $\tau$. Then the function $\phi(t+x_3)$ is a solution to the wave equation in the half-space with the Neumann data $g_{1,\phi}(x',t) = \phi'(t)$.

**Theorem 8.5.1** (Uniqueness of Discontinuity Surface). *Assume that*

$$-1 < k_j < 0, \qquad d_j < 0. \tag{8.5.1}$$

*If solutions $u_j$ to the initial boundary value problem* (8.0.1)–(8.0.3) *with $k = k_j$, $D = D_j$, and the Neumann data $g_1 = g_{1,\phi}$ on $\gamma_0 \times \{0,\tau\}$ for some $\tau$ satisfy the condition*

$$u_1 = u_2 \text{ on } \gamma_0 \times (0,T), \tag{8.5.2}$$

*then $D_1 \cap U = D_2 \cap U$.*

PROOF. Our first claim is that

$$u_1 = u_2 \text{ on } Q_T \backslash (Q_1 \cup Q_2), \tag{8.5.3}$$

where $Q_T$ is $\{0 < t + x_3, t - x_3 < T, |x'| < R, x_3 < 0\}$. To prove this, we denote by $\mathrm{Tr}(x^0, t^0)$ the triangle $\{t^0 + x_3^0 < t + x_3, t - x_3 < t^0 - x_3^0, x_3 < 0\}$, which is a translated and scaled triangle Tr from Lemma 3.4.6. Observe that $(x^0, t^0)$ is a vertex of this triangle. Let $(x^0, t^0)$ be a point in $Q_T \backslash (\overline{Q}_1 \cup \overline{Q}_2)$. Then there is positive $\epsilon$ such that the set $\{x' : |x' - x^{0'}| < \varepsilon\} \times \mathrm{Tr}(x^0, t^0)$ is contained in $Q_T \backslash (\overline{Q}_1 \cup \overline{Q}_2)$. On this set the function $u = u_2 - u_1$ satisfies the wave equation, and it has zero Cauchy data on the part of the boundary of this set contained in $\{x_3 = 0\}$. By Lemma 3.4.7 we have $u = 0$ on this set and hence at the point $(x^0, t^0)$.

Let us assume that the claim of Theorem 8.5.1 is wrong. After (possible) relabeling we can then assume that there is a point $y(0) \in (\partial D_2 \backslash \overline{D}_1) \cap U$. Since $\partial D_2$ is a Lipschitz surface, it has an (exterior) normal almost everywhere, so we can in addition assume that there is an exterior normal $\nu$ at $y(0)$. Let $y(\delta) = y(0) + \delta\nu$.

Let us introduce the functions

$$v(x,t) = \phi(t + x_3), v^*(x,t;y,\tau) = |x-y|^{-1}\phi(-A_0(t-\tau) - |x-y|),$$

where $\phi$ is the function from the beginning of this section and $A_0 = (1+k_2)^{1/2}$ is the speed of propagation inside $D_2$. We have

$$v = 0 \text{ when } t + x_3 < 0$$
$$v^* = 0 \text{ outside } \mathrm{con}_{A_0}(y,\tau). \tag{8.5.4}$$

Moreover, one can check that

$$(\partial_t^2 - \Delta)v = 0,$$
$$(8.5.5) \qquad (\partial_t^2 - (1 + k_2)\Delta)v^* = 0 \text{ when } x \neq y.$$

We have the following orthogonality relation.

**Lemma 8.5.2.** *Under the conditions of Theorem* 8.5.1*, we have*

$$(8.5.6) \qquad \int_{Q_2} k_2 \nabla v \cdot \nabla v^* = 0$$

*for any* $v^* = v^*(; y(\delta), (|y_3(0)| + \delta)/A_0 + \varepsilon)$ *when* $\delta, \varepsilon(\delta)$ *are positive and small.*

PROOF. Subtracting equations (8.0.1) for $u_2$ and $u_1$ and letting $u = u_2 - u_1$, we obtain

$$(\partial_t^2 u - \operatorname{div}((1 + k_2\chi(Q_2))\nabla u) = \operatorname{div}((k_2\chi(Q_2) - k_1\chi(Q_1))\nabla u_1).$$

From (8.5.3) we have $u = 0$ on $Q_T \backslash (Q_1 \cup Q_2)$. In addition, when $\delta, \varepsilon$ are small we have $\nabla v \cdot \nabla v^* = 0$ outside a small neighborhood of the point $Y(0) = (y(0), |y_3(0)|)$ on $Q_2$. To convince yourself of this observe that the intersection of the backward cone with the half-space $\{t + x_3 > 0\}$ will be in a small neighborhood of $Y(0)$ for small $\delta$, since the surface of this cone is "steeper" than the boundary of the half-space (because the speed of propagation inside $Q_2$ corresponding to this cone is less than outside). Using the definition of $y(\delta)$ and a continuity argument, we can find small $\epsilon$ to guarantee that the intersection of the cone, the half-space, and $Q_2$ is simultaneously not empty and in a small neighborhood of $Y(0)$.

According to the definition (8.0.5) of the generalized solution we have

$$(8.5.7) \qquad \int_V (\partial_t u \partial_t \psi - (1 + k_2\chi(Q_2))\nabla u \cdot \nabla \psi) = -\int_V k_2 \nabla u_1 \cdot \nabla \psi$$

for any test function $\psi \in C_0^\infty(V)$. Since $u = 0$ on $V \backslash Q_2$, we can replace $1 + k_2\chi(Q_2)$ by $1 + k_2$ and use as $\psi$ on $C^\infty(V)$-function that is zero near $\partial V \cap Q_2$; in particular, $\psi = v^*$ with small $\delta, \varepsilon$. Integrating by parts on the left side of (8.5.7) to apply all partial differentiations to $\psi$ and exploiting that $v^*$ solves the wave equation (8.5.5) on $V \cap Q_2$, we conclude that the left side of (8.5.7) is zero. Therefore, we have obtained (8.5.6) with $u_1$ instead of $v$. Our choice of the lateral Neumann data guarantees that $v_1$ has the same Neumann data as $v$ near $\Gamma \times \{0\}$. Observing that by condition (8.5.1) the speed of propagation outside $Q_1$ is greater than that inside, we conclude that $v = u_1$ in $V$.

The proof is complete. □

We return to the proof of Theorem 8.5.1.

From the definition of $v, v^*$ and the equality

$$\partial_3 |x - y(\delta)| = |x - y(\delta)|^{-1}(x_3 - y_3(\delta))$$

we have

$$
\begin{aligned}
-\nabla v \cdot \nabla v^* &= -\phi'(t + x_3)\partial_3 v^* \\
&= \phi'(t + x_3)(|x - y(\delta)|^{-3}\phi(-A_0(t - \tau) - |x - y(\delta)|)| \\
&\quad + |x - y(\delta)|^{-2}\phi'(-A_0(t - \tau) - |x - y(\delta)|))(x_3 - y_3(\delta)),
\end{aligned}
$$

due to the assumption $\phi' > 0$. Besides, $x_3 - y_3(\delta) > 0$ when $x \in Q_2 \cap V$, provided that $\delta$ and $\varepsilon$ are small. Hence

$$
-\int_{Q_2} k_2 \nabla v \cdot \nabla v^* > 0 \tag{8.5.8}
$$

when $\delta$, $\varepsilon$ are small.

For small $\delta$ and $\varepsilon$, by Lemma 8.5.2 we have the equality (8.5.6), which contradicts the inequality (8.5.8). This contradiction shows that the assumption $Q_1 \cap U \neq Q_2 \cap U$ is wrong.

The proof is complete. □

In geophysics one is also interested in spherical waves $\phi(t + |x - a|)$ instead of plane waves. A minor modification of Theorem 8.5.1 gives a sharp uniqueness result in this case. Let a function $\phi$ satisfy the same conditions as in Theorem 8.5.1. Then $\phi(t + |x|)$ is a solution to the wave equation in the half-space $\{x_3 < 0\}$ with the Neumann data $h_{\phi s}(x', t) = \phi'(t + |x'|)|x|^{-1}x_3$.

Let us introduce the spherical uniqueness set $U_s = \{|x| < T/2, |x'| < R\}$.

**Theorem 8.5.3$_s$.** *Let condition* (8.5.1) *be satisfied. If for solutions $u_j$ of the initial boundary value problem* (8.0.1)–(8.0.3) *with $k = k_j$, $D = D_j$, and $g_1 = h_{\phi s}$ on $\Gamma_0 \times (0, \tau)$ for some $\tau$ we have the equality* (8.5.2)*, then $D_1 \cap U_s = D_2 \cap U_s$.*

**Exercise 8.5.3.** Give a proof of Theorem 8.5.1$_s$.

We expect that the proof of Theorem 8.5.1 can be repeated with the natural changes when applying sharp uniqueness of the continuation result, considering the intersection of two cones instead of the cone and the half-plane, and proving the inequality (8.5.8).

We think that the scheme of the proof of Theorem 8.5.1 can be transformed into an efficient algorithm of numerical reconstruction. To do so one considers one (unknown) equation (8.0.1), $a = 1 + k\chi(D)$, and makes use of solutions $v^*$ of the wave equation to form a functional similar to the integral (8.5.8). This functional can be found from the boundary measurements and the continuation of the wave field. For some simple geometries of $D$ this continuation is not needed. Then one can change $T$ and calculate this functional. The first time it is positive, the wave strikes $\partial D$. By using spherical waves from different sources $x^0$, one can calculate distances from $x^0$ to $\partial D$ and recover $D$.

In the paper [Is11] there is a similar result for the classical elasticity system. Hansen [H] considered a linearized variant of this problem. Rakesh [Ra] made use of the study of propagation of singularities after reflection from $\partial D$ to recover convex $D$ entering the acoustic equation

$$(1 + k\chi(D))\partial_t^2 - \operatorname{div}((1 + k\chi(D))\nabla u) = 0.$$

The coefficient of this equation involves discontinuity similar to that under the consideration, but the speed of the propagation is 1 everywhere.

## 8.6 Open problems

As in previous chapters we will list some outstanding research problems.

**Problem 8.1** (Two Speeds of Propagation). Give sufficient conditions for uniqueness of the coefficients $a_{01}(x), a_{02}(x) \in C^2[0, +\infty)$ of the fourth-order hyperbolic equation

$$(a_{01}\partial_t^2 - \partial_x^2)(a_{02}\partial_t^2 - \partial_x^2)u = 0 \text{ in } (0, +\infty) \times (0, T)$$

with zero initial conditions $u = \cdots = \partial_t^3 u = 0$ on $(0, 1) \times \{0\}$ and some lateral boundary data (e.g., $u = g_0$, $\partial_x u = g_1$ on $\{0\} \times (0, T)$ with prescribed $g_j$ satisfying the conditions of the type (8.1.9)) when one is given additional lateral data (e.g., $\partial_x^2 u$, $\partial_x^3 u$ on $\{0\} \times (0, T)$).

One can obtain simple conclusions from the results of Section 8.1.1. Say, if the lateral data are $u = t^2$, $\partial_x^2 u = 0$ on $\{0\} \times (0, T)$ and one is given the additional lateral data $\partial_x u$, $\partial_x^3 u$ on $\{0\} \times (0, T)$ and $a_{02}(0)$, $\partial_x a_{02}(0)$ are given as well, one obtains for the function $v = (a_{02}\partial_t^2 - \partial_x^2)u$ the second-order equation and data that guarantee uniqueness of $a_{01}$ due to Corollary 8.1.6. However, there are difficulties with proving uniqueness of $a_{02}$ (and of coefficients of more general higher-order equations as well as hyperbolic systems of first order). The main source of these difficulties is the presence of many speeds of propagation.

**Problem 8.2.** In the problem about identification of the coefficient $c$ in section 8.1 obtain sufficient/necessary conditions for a function $g(t)$, $0 < t < T$, to be the data of the inverse problem.

Observe that the first necessary conditions in the inverse problem for one-dimensional hyperbolic equations have been obtained by M. Krein [Kr], where there are no proofs. Later on in a similar problem, Symes [Sym] obtained sufficient conditions that are quite close to being necessary as well. Loosely speaking, these conditions mean that for any $C$ the kernel $C - g_0(s - t)$ is positively defined. Such conditions on $g_0$ are desirable, in particular for improving numerics, because the integral equation (8.1.4) has a strong (in fact Riccati-type or square) nonlinearity, and without proper conditions on $g$ its solution (which exists for small $T$) blows up for larger $T$. on $\partial\Omega \times (0, T)$ uniquely determine $b_0$ on $\Omega$, provided that $\operatorname{diam}\Omega < T$.

Let us consider the initial value problem

$$u + cu = 0 \text{ in } \Omega \times (0, T),$$
$$u = \partial_t u = 0 \text{ on } \Omega \times \{0\}, \qquad \partial_n u = g_1 \text{ on } \partial\Omega \times (0, T),$$

where $\Omega = \{x_n > 0\}$ and $g_1(x_1, \ldots, x_{n-1}, t) = 1$.

**Problem 8.3.** Prove uniqueness of the coefficient $c = c(x)$ in $\{x_n < T/2\}$ given $u$ on $\partial\Omega \times (0, T)$.

There is a similar question when $g_1 = \delta$ (the delta function with the pole at the origin) that looks even more difficult.

**Problem 8.4.** Is it possible recover all three scalar coefficients $a = a(x) \in C^1(\overline{\Omega})$, $b_0 = b_0(x) \in C^1(\overline{\Omega})$, and $c = c(x) \in L_\infty(\Omega)$ of equation (8.0.1) (with $a_0 = 1, b = 0, f = 0$) given its lateral Neumann-to-Dirichlet map for large $T$?

It is supposed to be solved by the methods of boundary control (Section 8.4). Observe that there is only an assumption on smoothness and positivity of $a$.

**Problem 8.5.** Obtain uniqueness results from single boundary measurements for some anisotropic Maxwell's and elasticity systems similar to results of Imanuvilov, Isakov, and Yamamoto [IIY] for classical isotropic dynamical elasticity system.

Difficulties here are in particular due to absence of Carleman estimates. Partial results are obtained by Belishev, Isakov, Pestov, and Sharafutdinov [BeIPS].

**Problem 8.6.** Prove uniqueness of all three time independent coefficients $\rho, \lambda, \mu$ of the dynamical Lame system with given lateral elastic Dirichlet-to-Neumann map.

So far there are partial results due to Rachele [R] under complicated geometrical assumptions on speeds of propagation.

**Problem 8.7.** Is it possible to remove the assumption (8.5.1) of Theorem 8.5.1 that tells that the speed of propagation inside $D$ is less then that outside?

# 9

# Inverse parabolic problems

## 9.0 Introduction

In this chapter we consider the second-order parabolic equation

(9.0.1) $$a_0 \partial_t u - \mathrm{div}(a \nabla u) + b \cdot \nabla u + cu = f \text{ in } Q = \Omega \times (0, T),$$

where $\Omega$ is a bounded domain the space $\mathbb{R}^n$ with the $C^2$-smooth boundary $\partial\Omega$. In Section 9.6 we study the nonlinear equation

(9.0.1n) $$a_0(x, u)\partial_t u - \Delta u + c(x, t, u) = 0 \quad \text{in } Q.$$

We prescribe the initial condition

(9.0.2) $$u = u_0 \text{ on } \quad \Omega \times \{0\}$$

and the lateral boundary condition

(9.0.3) $$u = g_0 \quad \text{on } \partial\Omega \times (0, T).$$

Here $a$ is a symmetric strictly positive matrix function with entries in $L_\infty(Q)$; $b$ is a (real-valued) vector function with the same regularity properties; and $a_0$, $c$ are (real-valued) $L_\infty(Q)$-functions, $a_0 > \varepsilon > 0$.

For these parabolic equations there are convenient anisotropic functional spaces $H_{2l,l;p}(Q)$ and $C^{2l,l}(\overline{Q})$, which we will describe below. The first one is formed of all functions $u(x, t)$ with finite norm

$$\|u\|_{2l,l;p}(Q) = \sum \|\partial_t^k \partial_x^\alpha u\|_p(Q)$$

where the sum is over $k, \alpha$ with $2k + |\alpha| \leq 2l$, $l$ is integer, and the second one of all functions $u(x, t)$ with the finite norm

$$|u|_{2l,l}(Q) = \sum |\partial_t^k \partial_x^\alpha u|_{2\lambda,\lambda}(Q),$$

where the sum is over the same set of indices, $\lambda = l - [2l]/2$. Here $|v|_{2\lambda,\lambda}(Q) = \sup |v(x + h, t + s) - v(x, t)|/(|h|^{2\lambda} + |s|^\lambda)$ over $(x + h, t + s), (x, t) \in Q$.

These are Banach spaces, and if $p = 2$, the first one is a Hilbert space with respect to an equivalent norm. Since we are interested in discontinuous coefficients $a$ of equation (9.0.1) a classical solution does not necessarily exist, so we recall the following standard definition of a generalized solution $u \in C([0,T]; H_{(1)}(\Omega))$ to the initial boundary value problem (9.0.1), (9.0.2), (9.0.3) with $f = f_0 + \operatorname{div} f^\bullet$, $f_0, f^\bullet \in L_2(Q)$ as a function satisfying the integral identity

$$\int_Q (-u\partial_t(a_0 v) + a\nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v)dQ$$
$$= \int_Q (f_0 v - f^\bullet \cdot \nabla v)dQ + \int_\Omega a_0 u_0 v(,0)$$

(for all (test) functions $v \in H_{1,2}(Q)$ that are zero on $\Omega \times \{T\}$ and on $\partial\Omega \times (0,T)$) as well as the lateral boundary condition (9.0.3). Here we have assumed that $\partial_t a_0 \in L_\infty(Q)$.

We will describe some basic solvability and regularity results about the direct problem (9.0.1)–(9.0.3) in the following theorem.

**Theorem 9.1.** *(i) (Weak Solutions) Assume that $f_j \in L_2(Q), u_0 \in L_2(\Omega)$, $g_0, \partial_t g_0 \in L_2((0,T); H_{(1/2)}(\partial\Omega))$.*

*Then there is a unique (generalized) solution $u \in C([0,T]; H_{(1)}(\Omega))$ to the initial boundary value problem* (9.0.1)–(9.0.3).

*(ii) (Regularity) If $f_j \in L_\infty(Q)$ and $u \in C^{\lambda,\lambda/2}(\Gamma)$, where $\Gamma$ is an (open) part of $\overline{\partial Q_T}$, $0 < \lambda < 1$ then $u \in C^{\mu,\mu/2}(Q \cup \Gamma)$ with some $\mu \in (0,1)$.*

*If $\partial\Omega \in C^{2+\lambda}$, $a_0, a^{jk}, \partial_m a^{jk}, b_k, c, f_0 \in C^{\lambda,\lambda/2}(\overline{Q}^0)$, $f_j = 0$, $u_0 \in C^{2+\lambda}(\Omega \cap \overline{Q}^0)$, $g_0 \in C^{2+\lambda,1+\lambda/2}(\partial\Omega \times [0,T] \cap \overline{Q}^0)$, where $Q^0$ is a subdomain of $Q$ and the following compatibility conditions are satisfied: $u_0 = g_0$, $\partial_t g_0 + Au_0 = f_0$ on $\partial\Omega \times \{0\}$, then $u \in C^{2+\lambda,1+\lambda/2}(Q^0 \cup \Gamma^0)$, where $\Gamma^0$ is an open part of $\partial Q \cap \overline{Q}^0$.*

*Moreover, under the mentioned Hölder regularity conditions on the coefficients, for any subdomain $Q^1$ of $Q^0$ with positive distance to $\partial Q^0 \cap Q$ there is a constant $C$ depending on this distance such that*

$$|u|_{2+\lambda,1+\lambda/2}(Q^1) \le C(|f|_{\lambda,\lambda/2}(Q^0) + |g_0|_{2+\lambda,1+\lambda/2}(\partial\Omega \times [0,T] \cap \partial Q^0)$$
$$+ |u_0|_{2+\lambda}(\Omega \times \{0\} \cap \partial Q^0) + \|u\|_\infty(Q^0))$$

*and when only $a_0, a, \nabla a, b, c \in C(\overline{Q})$ and $p \ne 3/2$, then*

$$\|u\|_{2,1;p}(Q^1) \le C(\|f\|_p(Q^0)$$
$$+ \|g_0\|_{2-2/p,1-1/p;p}(\partial\Omega \times [0,T] \cap \partial Q^0)$$
$$+ \|u_0\|_{2-2/p,p}(\Omega \times \{0\} \cap \partial Q^0) + \|u\|_p(Q^0)).$$

*If $Q^0 = Q$, the terms $\|u\|_\infty(Q)$ and $\|u\|_p(Q)$ can be dropped.*

Here $\partial Q_T$ is $\partial Q \cap \{t < T\}$. Essentially, these results are proven in the book of Ladyzhenskaya, Solonnikov, and Ural'tseva [LSU]. Part (i) follows from Theorem 4.2 of [LSU] when we reduce our case to the case of zero lateral data by using a function $U$ that is in $L_2((0, T); H_{(1)}(\Omega))$ together with $\partial_t U$ and that coincides with $g_0$ on $\partial\Omega \times (0, T)$. The existence of such a function follows from extension theorems for Sobolev spaces. Observe that $U \in C([0, T], L_2(\Omega))$. Then we let $u = U + u_0$ and make use of the results of [LSU] for $u_0$. Local bounds are given in [LSU, chapter 4, section 10].

The regularity part of Theorem 9.1 in particular implies continuity of solutions of parabolic equations with discontinuous coefficients, provided that the initial and lateral boundary data are Hölder continuous and the source term is measurable and bounded.

Using the known definition of a generalized solution to equation (9.0.1) with lateral Neumann data $a\nabla u \cdot \nu = \Lambda u$ on $\partial_x Q$ ([LSU, p. 168]), we obtain the useful identity

$$\int_Q (-u\partial_t(a_0 v) + a\nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v)$$
$$= \int_Q (f_0 v - f^\bullet \cdot \nabla u) + \int_{\partial\Omega\times(0,T)} \Lambda uv - \int_\Omega a_0 u_0 v(,0) \tag{9.0.4}$$

for all test functions $v \in H_{1,2}(Q)$ that are equal to zero on $\Omega \times \{T\}$. This identity makes sense for generalized solutions $u$ from Theorem 9.1(i).

Solutions to linear parabolic equations have remarkable positivity properties, which we will formulate as follows.

**Theorem 9.2** (The Positivity Principle). *If $f_j = 0$, $f_0 \geq 0$ on $Q$, $g_0 \geq 0$ on $\partial\Omega \times (0, T)$, $u_0 \geq 0$ on $\Omega$, then $u \geq 0$ on $Q$. In addition, if $u(x, t) > 0$, then $u(y, s) > 0$ when $y \in \Omega, t < s \leq T$.*

This result follows directly from the maximum principles when the coefficients, the source term, and the initial and boundary data are (Hölder) smooth. We refer to the book of Friedman [Fr, chapter 2]. In the general case we can approximate the coefficients, source term, and boundary data by smooth nonnegative data and pass to the limit by using known stability theorems ([LSU, Theorem 4.5, p. 166]).

**Exercise 9.3** (Maximum Principle). Prove that if in equation (9.0.1) $f_j = 0$ and $c \geq 0$ in $Q$, then $u \leq \max u$ over $\partial Q_T$ provided $\max u \geq 0$.

{*Hint*: let $W = \max u$ over $\partial Q_T$ and apply Theorem 9.2 to difference $W - u$.}

**Exercise 9.4.** Let $u \in C^{2,1}(Q) \cap C(\overline{Q})$ be a solution to the nonlinear equation (9.0.1n). Assuming that $c(x, t, u)u \geq 0$ for all $u$ and $U_- \leq u \leq U_+ (U_-, U_+$ and negative and positive constants) on $\partial Q_T$, prove that $U_- \leq u \leq U_+$ on $Q$.

{*Hint*: to prove that $w = U_+ - u \geq 0$, observe that the nonlinear operator (9.0.1n) at $U_+$ is $c(x, t, U_+) \geq 0$. Subtract the nonlinear parabolic equations for $U_+$ and $u$ to obtain for $w$ a linear parabolic equation with the source term $c(x, t, U_+)$ and apply Theorem 9.2.}

In Sections 9.3 and 9.5 we will make use of analyticity in time of solutions of parabolic problems with time-independent coefficients and of stabilization of solutions of equations with maximum principle to solutions that do not depend on $t$. We are going to formulate related results about direct problems. Let $A = -\operatorname{div}(a\nabla) + b \cdot \nabla + c$.

**Theorem 9.5** (Analyticity with Respect to $t$). *Assume that the coefficients of A do not depend on t; $a_0, \nabla a \in C(\overline{\Omega}); b, c \in L_\infty(\Omega)$; the right side of* (9.0.1) $f = 0$*; and the Dirichlet data $g_0 \in C^2(\partial\Omega \times \mathbb{R}^+)$ are (complex) analytic in a sector containing the ray $(T - \varepsilon, \infty)$.*

*Then a solution to the first boundary value problem* (9.0.1), (9.0.2), (9.0.3) *is complex analytic with respect to t in a smaller sector. There is C depending only on $\Omega$, $H_{1,\infty}(\Omega)$-norms of $a_0, a, b, c$, and the ellipticity constant of A such that for a solution u to the parabolic boundary value problem* (9.0.1), (9.0.2), (9.0.3) *is with $T = \infty$, $f = 0$, $u_0 = 0$, and $g_0 \in H_{(1/2)}(\partial\Omega \times (0, \tau))$ that does not depend on $t \in (\tau/2, \infty)$, one has*

$$\|u(, t)\|_{(1)}(\Omega) \leq C\|e^{Ct} g_0)\|_{(1/2)}(\partial\Omega \times (0, \tau)), t \in S.$$

This result follows from the abstract theory of holomorphic semigroups of operators, described, for example, in the books of Krein [Kre] and Yosida ([Yo], section 9.10). Applicability of the basic results of the abstract theory follows from the bound of the resolvent of corresponding elliptic boundary value problems given in several papers. We refer to the most recent of them, by Colombo and Vespri [CoV], which contains an extensive bibliography.

**Theorem 9.6.** *Assume that the coefficients of A do not depend on t; $0 \leq c$, and $g_0$ does not depend on $t \in (\tau/2, \infty)$, $g_0 \in C^2(\partial\Omega \times \mathbb{R}^+)$, $\|g_0\|_{(1/2)}(\partial\Omega \times (0, \tau)) \leq 1$. Let $u^0$ be the solution to the Dirichlet problem $Au^0 = 0$ in $\Omega$, $u^0 = g_0(, \tau)$.*

*Then there are C and $\theta > 0$ depending only on $\Omega, \tau$, on the norms of the coefficients and of $\nabla a$ in $L_\infty(\Omega)$, and on the ellipticity constant of A such that*

$$\|u(, t) - u^0\|_\infty(\Omega) \leq Ce^{-\theta t}, \ \textit{when } \tau/2 < t.$$

This result is proven in the book of Friedman [Fr, pp. 158–161].

In inverse problems one is looking for one or several coefficients of equation (9.0.1) or for the source term $f$, when in addition to the initial and lateral boundary data (9.0.2), (9.0.3) we are given either the final data

$$u = u_T \text{ on } \omega \times \{T\} \tag{9.0.5}$$

where $\omega$ is an open subset of $\Omega$, or the lateral (Neumann) data

$$a\nabla u \cdot \nu = g_1 \text{ on } \gamma \times (0, T), \tag{9.0.6}$$

where $\gamma$ is a part of $\partial\Omega$.

## 9.1 Final overdetermination

In this section we will consider the inverse problem with additional data at the final moment of time $T$. It was introduced as early as in 1935 by Tikhonov [Ti], motivated by a geophysical interpretation (recovery of a geothermal prehistory from contemporary data). He studied the one-dimensional heat equation in the half-plane and proved the uniqueness of its (bounded) solution with prescribed lateral and final data. In the 1950s this backward parabolic problem was studied by Fritz John [Jo1], [Jo3]. We refer for more detail to Section 3.1. In particular, the backward parabolic problem is ill-posed in the sense of Hadamard. When both the initial and final data are given, it turns out also to be possible to find a coefficient of a parabolic equation. Moreover, the problem becomes stable but of course nonlinear. In this section we recall some results concerning this problem referring for additional results to the books [Is4], [PrOV] and to the papers of Isakov [Is6], and Prilepko and Solov'ev [PrS].

We will assume that $\partial\Omega$ is of class $C^{2+\lambda}$. In Theorem 9.1.1 $\lambda$ is any number in $(0, 1)$.

**Theorem 9.1.1.** *Let us assume that* $f = \alpha F$, *where* $F \in C^{\lambda}(\overline{\Omega})$, $\partial_t F = 0$ *in* $Q$, *and that the coefficients of equation* (9.0.1) *and the weight function* $\alpha$ *are given, that these coefficients, their* $t$*-derivatives,* $\alpha$, $\partial_1\alpha$ *are in* $C^{\lambda,\lambda/2}(\overline{Q})$, *and that* $\varepsilon_0 < \alpha$ *on* $\omega \times \{T\}$. *Let* $Q_\omega$ *be a subdomain of* $Q$ *such that* $\overline{Q}_\omega \subset \omega \cup \partial\Omega \times [0, T]$.

*Then there is a constant* $C$ *depending only on* $Q$, $\omega$, $Q_\omega$, $|\ |_{\lambda,\lambda/2}(Q)$*-norms of the coefficients of the parabolic equation* (9.0.1) *and of* $\alpha$ *as well as their* $t$*-derivatives, and on the positive number* $\varepsilon_0$ *such that any solution* $(u, F)$ *to the inverse source problem* (9.0.1), (9.0.2), (9.0.3), (9.0.5) *satisfies the inequality*

$$
\begin{aligned}
(9.1.1) \qquad |u|_{\lambda+2,\lambda/2+1}(Q_\omega) + |F|_\lambda(\omega) &\le C(|u_0|_{\lambda+2}(\Omega) \\
&+ |u_T|_{\lambda+2}(\omega) + |g_0|_{\lambda+2,\lambda/2+1}(\partial\Omega \times (0, T)) + \|F\|_\infty(\Omega)).
\end{aligned}
$$

PROOF. We can assume that $u_0 = 0$, $g_0 = 0$ (it can be achieved by subtracting a solution of the direct parabolic problem with the data $u_0$, $g_0$). We let $t = T$ in the equation (9.0.1) and use the final overdetermination (9.0.5) to get

$$
(9.1.2) \qquad F - \alpha^{-1}(, T)a_0(, T)\partial_t u(, T; F) = \alpha^{-1}(, T)A(, T)u_T,
$$

where $u(x, t; F)$ denotes the solution to the (direct) parabolic problem (9.0.1)–(9.0.3) with $f = \alpha F$ and $A$ the second-order elliptic operator formed from the terms of the left side of (9.0.1) not involving $\partial_t u$. Letting $v = \partial_u(; F)$ and differentiating (9.0.1) with respect to $t$ we obtain for $v$ the parabolic equation (9.0.1) with

$$
f = -\partial_t a_0 \partial_t u + \mathrm{div}(\partial_t a \nabla u) - \partial_t b \cdot \nabla u - \partial_t c u + \partial_t \alpha F.
$$

Using the bounds of Theorem 9.1 with any $p > 1$ we conclude that the $L_p(Q)$-norms of all partial derivatives in the expression for $f$ are bounded by $C\|F\|_\infty(\Omega)$. Using again the $L_p$ bounds of Theorem 9.1 with large $p$ and embedding theorems for Sobolev spaces we yield

$$|\partial_t u(;F)|_\lambda(\Omega) \le C\|v\|_{2,1;p}(Q) \le C^2\|F\|_\infty(\Omega).$$

Now the bound (9.1.1) for $F$ follows from (9.1.2) and then the bound for $u$ follows from interior Hölder estimates of Theorem 9.1. □

This result does not (and can not) imply uniqueness.

**Theorem 9.1.2.** *We assume that the coefficients of the parabolic equation (9.0.1) and the weight function $\alpha$ are given and that they satisfy the following conditions*

$$a, b \text{ do not depend on } t; 0 \le c,\ \partial_t c \le 0 \text{ on } Q; \partial_t a_0,\ \partial_t c,\ \alpha,\ \partial_t\alpha \in C^{\lambda,\lambda/2}(\overline{Q}),$$

$$(9.1.3) \qquad 0 \le \alpha, 0 \le \partial_t\alpha \text{ on } Q; 0 < \alpha \text{ on } \omega \times \{T\}.$$

*Then a solution $(u, F) \in C^{\lambda,\lambda/2}(\overline{Q}) \times C^\lambda(\Omega)$ to the inverse source problem (9.0.1)-(9.0.3), (9.0.5) with $f = \alpha F$, $F = 0$ on $\Omega \setminus \omega$ is unique.*

PROOF. Now we will give a short and complete proof of uniqueness based on the positivity principle, as first suggested by Prilepko and Solov'ev [PrS]. Since the inverse source problem is linear, it suffices to show that $u = 0$ in $Q$, provided that the boundary data (9.0.2), (9.0.3), (9.0.5) are zero.

Let us assume that $F$ is not zero on $\omega$. Let $F^+ = (F + |F|)/2$ and $F^- = (-F + |F|)/2$. Since $\alpha \ge 0$, we have $(\alpha F)^+ = \alpha F^+$, $(\alpha F)^- = \alpha F^-$. Since $F \in C^\lambda(\overline{\Omega})$, the functions $F^+, F^-$ have the same regularity. If $F^- = 0$, then $0 \le F$, and by the positivity principle $0 < u$ on $\Omega \times \{T\}$, which contradicts the homogeneous condition (9.0.5). Similarly, the case $F^+ = 0$ is not possible. Let $\omega^+ = \{x \in \omega : F^+(x) > 0\}$ and $\omega^- = \{x \in \omega : F^-(x) < 0\}$. Then both sets $\omega^+, \omega^-$ are nonempty open subsets of $\omega$.

Let $u^+, u^-$ be solutions to the parabolic problem (9.0.1), (9.0.2), (9.0.3) with zero initial and lateral boundary data and with the source term $\alpha F^+, \alpha F^-$. By Theorem 9.1 these solutions exist and are in $C^{2+\lambda,1+\lambda/2}(\overline{Q})$. By the positivity principle, $u^+, u^-$ are nonnegative. We claim that

$$(9.1.4) \qquad 0 < \partial_t u^+, 0 < \partial_t u^- \quad \text{on } Q \text{ and on } \Omega \times \{T\}.$$

By using finite differences with respect to $t$ and a priori estimates of solutions to parabolic problems, one can show that $w^+ = \partial_t u^+ \in C^{2+\lambda,1+\lambda/2}(\overline{Q})$. Differentiating equation (9.0.1) with respect to $t$ we get

$$a_0\partial_t w^+ - \operatorname{div}(a\nabla w^+) + b \cdot \nabla w^+ + (c + \partial_t a_0)w^+ = \partial_t\alpha F^+ - \partial_t c u^+ \ge 0 \text{ in } Q,$$

by conditions (9.1.3). From the equation for $u^+$ at $t = 0$ we have $w^+ = \partial_t u^+ = -A0 + \alpha a_0 F^+ \ge 0$ on $\Omega$. Obviously, $w = 0$ on $\partial\Omega \times (0, T)$. Applying Theorem 9.2 again, we obtain the first inequality (9.1.4). A proof of the second one is similar.

Since $u^+ \in C(\overline{\Omega})$ and is zero on $\partial\Omega \times \{T\}$, it has a maximum point. Due to (9.1.4) this maximum point is $(x^0, T)$. Since $F^+ = 0$ outside $\omega$ Theorem 9.2 forbids $x^0$ to be outside $\bar{\omega}^+$, so $x^0 \in \bar{\omega}$. Similarly, maximum of $u^-$ is achieved on

$\bar{\omega}^- \times \{T\}$. The homogeneous condition (9.0.5) implies that $u^+ = u^-$ on $\omega \times \{T\}$, so $(x^0, T)$ is also a maximum point of $u^-$ on $\Omega \times \{T\}$. Consequently, $x^0 \in \bar{\omega}^+ \cap \bar{\omega}^-$, $\nabla u^+(x^0, T) = 0$ and we from (9.0.1) we have

$$\operatorname{div}(a\nabla u^+)(x^0, T) = c(x^0, T)u^+(x^0, T) + a_0\partial_t u(x^0, T) > 0,$$

due to conditions (9.1.4) and to (9.1.5). But at an interior maximum point $x^0$ the left side can not be positive. The contradiction shows that the initial assumption was wrong, so $F = 0$. Then $u = 0$ by Theorem 9.1.

The proof is complete. □

**Corollary 9.1.3.** *Under conditions of Theorems 9.1.1 and 9.1.2 there is a constant $C$ depending on the same parameters as in Theorem 9.1.1 such that*

$$|F|_\lambda(\omega) \le C(|u_T|_{2+\lambda}(\omega) + |u_0|_{2+\lambda}(\Omega) + [g_0]_{2+\lambda}(\partial\Omega \times (0, T)))$$

This corollary follows by compactness-uniqueness argument (see the proof of Theorem 3.4.11) from Theorems 9.1.1, 9.1.2.

Theorems 9.1.1, 9.1.2 are easy to apply to identification of coefficients problems. The monotonicity condition (9.1.1) is somehow restrictive, especially in applications to identification of coefficients. It can be relaxed, to a certain inequality involving the first eigenvalue of the corresponding elliptic boundary value problem, but it can not be removed completely. Indeed, in the paper [Is6] there is a (quite complicated) counterexample that shows that the uniqueness claim is generally false if the condition $\partial_t\alpha > 0$ is not satisfied. In more detail, in $Q = (0, 1) \times (0, T)$ one can construct nonzero $C^\infty(\overline{Q})$-functions $u, \alpha, F$ with $\partial_t F = 0$ such that

$$\partial_t u - \partial_x^2 u = \alpha F, \text{ on } \partial Q, 0 < \alpha \text{ on } \overline{Q}.$$

There is no such counterexample in the following (nonlinear) problem on identification of coefficients.

Now, we are looking for the coefficient $c = c(x) \in C^\lambda(\overline{\Omega})$ of the parabolic initial boundary value problem (9.0.1)–(9.0.3) with $a_0 = a = 1, b = 0, f = 0, u_0 = 0$, and given $g_0 \in C^{2+\lambda,1+\lambda/2}(\partial\Omega \times [0, T])$, $g_0 = \partial_t g_0 = 0$ on $\partial\Omega \times \{0\}$ from the additional final overdetermination (9.0.5).

**Theorem 9.1.4.** *(i) (Uniqueness) If*

(9.1.5) $$0 \le g_0, 0 \le \partial_t g_0 \text{ (and not identically zero) on } \partial\Omega \times (0, T),$$

*then a solution $(u, c)$ of the inverse problem is unique.*

*(ii) (Stability) If $(u_1, c_1), (u_2, c_2)$ are solutions of the inverse problem with the data $g_1, u_{01}, g_2, u_{02}$ satisfying condition* (9.1.5) *and the additional condition* $0 < \varepsilon_0 < g_j$ *on $\partial\Omega \times \{T\}$, then*

(9.1.6) $$|c_2 - c_1|_\lambda(\Omega) + |u_2 - u_1|_{2+\lambda,1+\lambda/2}(Q) \le C(|g_{02} - g_{01}|_{2+\lambda,1+\lambda/2}(\partial\Omega \times (0, T)) + |u_{T2} - u_{T1}|_{2+\lambda}(\Omega)),$$

*where $C$ depends on the same parameters as in Theorem* 9.1.1, *on* $\varepsilon_0$, *and on the norms of* $g_{0j}$, $u_{Tj}$ *used on the right side of inequality* (9.1.6).

*(iii) (Existence) If in addition,* $u_T \in C^{2+\lambda}(\overline{\Omega})$ *satisfies the conditions*

$$-\Delta u_T + \partial_t v(, T) \le 0, 0 \le u_T \text{ on } \Omega,\, g_0 = u_T \text{ on } \partial\Omega \times \{T\}, \tag{9.1.7}$$

*where $v$ is a solution to the initial boundary value problem* (9.0.1)–(9.0.3) *with* $c = 0$, *then there is a solution* $(u, c) \in C^{2+\lambda,1+\lambda/2}(\overline{Q}) \times C^{\lambda}(\overline{\Omega})$ *to the inverse problem* (9.0.1)–(9.0.3), (9.0.5).

PROOF. (i) Using a possible translation with respect to $t$, one assumes that $g_0$ is not zero on any set $\partial\Omega \times (0, \tau)$. Observe that then

$$0 < u, 0 < \partial_t u \quad \text{on } Q. \tag{9.1.8}$$

Indeed, the first inequality follows immediately from the positivity principle and our condition on the data of the problem. To prove the second inequality we differentiate equation (9.0.1) with respect to $t$ and apply the positivity principle again.

Now, if $u_2, c_2$ and $u_1, c_1$ are solutions to the inverse problem, then subtracting the equations for them and letting $u = u_2 - u_1, \alpha = u_2, F = c_1 - c_2$, we obtain the inverse source problem discussed in Theorem 9.1.2. Condition (9.1.3) is satisfied due to (9.1.8). By this theorem, $u = 0$, $F = 0$, which completes the proof of uniqueness.

(ii) To prove stability estimate (9.1.6) we can repeat the above subtraction procedure and apply the estimate of Theorem 9.1 and Corollary 9.1.3.

(iii) The proof we will suggest is constructive and employs the (mononote) iterations $u(; j), c(; j-1)$ defined as follows: $u(; j)$ solves the parabolic boundary value problem (9.0.1)–(9.0.3) with

$$\begin{aligned} &c = c(; j-1), \\ &c(;0) = 0, c(; j) = (\Delta u_T - \partial_t u( , T; j))/u_T. \end{aligned} \tag{9.1.9}$$

From Theorem 9.1 it follows that $u(; j)$ exists and is in $C^{2+\lambda,1+\lambda/2}(\overline{Q})$.

By using the positivity principle and conditions (9.1.5),(9.1.7), we will show that

$$u( ; j+1) \le u(; j), \partial_t u( ; j+1) \le \partial_t u( ; j), \tag{9.1.10}$$

$$c( ; j-1) \le c(; j). \tag{9.1.11}$$

Indeed, subtracting the equations for $u(; j+1)$ and $u(; j)$ and letting $w$ be the difference of these functions as above, we obtain for $w$ the parabolic initial boundary value problem (9.0.1)–(9.0.3) with zero initial and lateral boundary data and with the source term $u(; j)(c(; j-1) - c(; j))$. As in part (i), we have $0 \le u(; j)$, so (9.1.11) implies that the source term is positive, and therefore by the positivity principle, $0 \le w$. Differentiating the equations with respect to $t$, we obtain the second inequality (9.1.10). Now it remains to prove (9.1.11), which can be done by using induction in $j$. When $j = 0$, the result follows from definition (9.1.9). When $j = 1$, it follows from condition (9.1.7). If (9.1.11) is valid for $j \ge 1$, then from the definition of $c(;j)$ in (9.1.9) and from the second inequality (9.1.10), we obtain (9.1.11) for $j+1$.

Since $u(;j)$, $\partial_t u(;j)$ are solutions to an initial boundary value parabolic problem with nonnegative data, from the positivity principle we conclude that these functions are nonnegative on $Q$. Using, in addition, the definition (9.1.9) of $c(;j)$, we obtain

$$0 \le u(\,;j), 0 \le \partial_t u(\,;j), \quad c(\,;j) \le \Delta u_T/u_T. \tag{9.1.12}$$

From inequalities (9.1.10), (9.1.11), and (9.1.12) we conclude that the sequence of functions $c(\,;j-1)\partial_t u(;j)$ is bounded in $L_\infty(Q)$. Considering the initial boundary value problem (9.0.1)–(9.0.3) for $\partial_t u(\,;j)$ again, transferring the term $c(\,;j)\partial_t u(\,;j)$ onto the right side of the equation, and treating it as a source term from the $L_p(Q)$-estimates of Theorem 9.1, we conclude that the sequence of functions $\partial_t u(\,;j)$ is bounded in $H_{2,1;p}(Q)$. Therefore, by embedding theorems it is bounded in $C^{\lambda,\lambda/2}(\overline{Q})$. According to definition (9.1.9), the sequence of the coefficients $c(\,;j)$ is bounded in $C^\lambda(\overline{\Omega})$. We can make use of the Hölder estimates of Theorem 9.1 applied to the forward problem for $u(\,;j)$ to conclude that

$$|u(\,;j)|_{2+\lambda,1+\lambda/2}(Q) + |c(\,;j)|_\lambda(\Omega) \le C.$$

According to (9.1.10), (9.1.11), the sequences of functions $u(\,;j)$, $c(\,;j)$ are monotone with respect to $j$, and they are bounded in the corresponding Hölder functional spaces by the last estimate. Hence, they converge to some functions $u$, $c$ in $C^{2,1}(\overline{Q})$, $C(\overline{\Omega})$. From our bounds it follows that $u \in C^{2+\lambda,1+\lambda/2}(\overline{Q})$, $c \in C^\lambda(\overline{\Omega})$. Passing to the limit in the equations for $u(;j)$ we obtain equation (9.0.1) for $u$ with the coefficient $c$.

Passing to the limit in the definition (9.1.9) of $c(;j)$, we obtain the equality $\partial_t u - \Delta u_T + cu_T = 0$ on $\Omega \times \{T\}$. Using the differential equation for $u$ at $t = T$, we then obtain $-\Delta(u - u_t) + c(u - u_T) = 0$ on $\Omega \times \{T\}$. From the compatibility condition (9.1.7) we have $u - u_T = 0$ on $\partial\Omega \times \{T\}$. Since $c \ge 0$, the elliptic equation for $u - u_T$ satisfies the maximum principle, so $u - u_T = 0$ on $\Omega \times \{T\}$.

The proof is complete. □

A similar result is valid for principal coefficients $a_0 = a_0(x)$ of the problem (9.0.1)–(9.0.3), (9.0.5) with $a = 1, b = 0, c = 0, f = 0$, and $u_0 = 0$. We will formulate it as the following exercise.

**Exercise 9.1.5.** (i) Prove that under the conditions $0 < \partial_t g_0, 0 \le \partial_t^2 g_0$ a solution $(u, a_0)$ to the identification problem is unique.

(ii) Prove that under the additional condition

$$0 < \Delta u_T \le a_\star \partial_t v(, T) \text{ on } \overline{\Omega},\ g = u_T \text{ on } \partial\Omega \times \{T\},$$

where $v$ is a solution to the problem (9.0.1)–(9.0.3) with $a_0 = a_\star$, a solution $(u, a_0)$ to the identification problem exists and can be found by the monotone iterations

$$a_0(;0) = a_\star, a_0(;j) = \Delta u_T/\partial_t u(, T; j),$$

where $u(;j)$ is a solution to the parabolic problem (9.0.1)–(9.0.3) with the coefficient $a_0 = a_0(;j-1)$.

In thermal applications the assumption $u_0 = 0$ is not natural physically, but in the equations with no zero-order terms (as considered in Exercise 9.1.5), in describing propagation of heat and other diffusion phenomena one can consider constant initial conditions and then subtract this constant from a solution to the parabolic problem. This subtraction does not affect the assumptions of this exercise.

The inverse problems for equations in divergence form are not easy to handle. However, one can obtain uniqueness results in the one-dimensional case for the problem of finding $(u, a)$ entering the initial boundary value problem

$$\partial_t u - \partial_x(a\partial_x u) = 0 \quad \text{on } (0,1)\times(0,T),\ u = 0 \text{ on } (0,1)\times\{0\},$$
$$\partial_x u = 0 \text{ on } \{0\}\times(0,T),\ u = g_0 \text{ on } \{1\}\times(0,T),$$

with the final overdetermination $u = u_T$ on $(0,1)\times\{T\}$. We assume that $g_0, \partial_t g_0 \in C^{1+\lambda/2}([0,T])$ and that $g_0(0) = \cdots = \partial_t^2 g_0(0) = 0$.

**Exercise 9.1.6.** Assuming that $0 \le \partial_t g_0$, $0 \le \partial_t^2 g_0$ on $(0,T)$, prove that a solution $(u,a)$, $a = a(x) \in C^\lambda([0,1])$, of the inverse problem is unique.

{*Hint*: Introduce the new unknown function $w = a\partial_x u$, derive for it a nondivergent parabolic equation, and apply to the new inverse problem the suitably modified method of proof of Theorem 9.1.2 as it has been done in the proof of Theorem 9.1.4. One will need maximum principles for normal derivatives similar to Theorem 4.2 for elliptic equations.}

## 9.2 Lateral overdetermination: single measurements

While stable and computationally feasible, inverse problems with final overdetermination do not very often reflect interesting applied situations when one is given only additional lateral data. In many cases, the results of all possible lateral boundary measurements are available. However, even in this case, we have a severely ill-posed inverse problem that is challenging both theoretically and numerically. We start our discussion of lateral overdetermination with single boundary measurements; i.e., we are given one set of lateral boundary data $\{g_0, g_1\}$ on the lateral boundary $\partial\Omega\times(0,T)$ or on a part of it.

We will make use of the known transform

(9.2.1) $$u(x,t) = (\pi t)^{-1/2}\int_0^\infty \exp(-\tau^2/(4t))u^*(x,\tau)d\tau.$$

Observe that

(9.2.2) $$u(,t) = v(,0,t),$$

where $v = v(\tau, t)$ is the solution to the following standard parabolic problem:

$$\begin{aligned} \partial_t v - \partial_\tau^2 v &= 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}_+, \\ v &= u^* \text{ on } \mathbb{R}_+ \times \{0\}, \\ \partial_\tau v &= 0 \text{ on } \{0\} \times \mathbb{R}_+. \end{aligned} \tag{9.2.3}$$

This follows from the known basic representation of the solution of the one-dimensional Cauchy problem, provided that we extend the initial data $u^*$ onto $\mathbb{R}$ as an even function:

$$v(\tau, t) = (4\pi t)^{-1/2} \int_0^\infty (e^{-(\tau-\theta)^2/(4t)} + e^{-(\tau+\theta)^2/(4t)}) u^*(\theta) d\theta. \tag{9.2.4}$$

We observe that when $\partial_\tau^2 u^* \in L_2(0, R)$ for any positive $R$ and $\partial_\tau u^*(0) = 0$, one can differentiate the equation and the initial conditions of the extended Cauchy problem (9.2.3). Since from the heat equation $\partial_t v$ solves the Cauchy problem with the initial data $\partial_\tau^2 u^*$, we conclude that the operator (9.2.1) transforms $\partial_\tau^2 u^*$ into $\partial_t u$. The transformation $u^* \to u$ is very stable, while the inverse one is quite unstable (as a solution of the lateral Cauchy problem).

**Theorem 9.2.1.** *Let*

$$\|u^*\|_\infty(0, T) \le T^2 M^2 e^{MT}, \ \|\partial_\tau u^*\|_\infty(0, T) \le T M^2 (1 + T^2 M^2) e^{MT} \tag{9.2.5}$$

*and* $\varepsilon = \|u(, 0)\|_2(T/8, T)$.

*Then there is $C$ depending only on $T, T^*, \delta$ such that*

$$|u^*(\tau)| \le C C(\delta, M)((-\log\varepsilon)^{-1} + M(-\log\varepsilon)^{-1+\delta}) \tag{9.2.6}$$

*where* $C(\delta, M) = \sigma^{1/2} e^{\sigma^2 M^2 T + 4MT^*}$.

We outline a proof based on the following auxiliary results.

**Lemma 9.2.2.** *Under the conditions (9.2.5) for a solution $v$ to the problem (9.2.3) we have*

$$|v(\tau, t)| \le 2e^{4M^2 t + 2M\tau}, \ |\partial_\tau v(\tau, t)| \le 4.2 M e^{4M^2 t + 2M\tau}.$$

PROOF. First, we observe the elementary inequality

$$(4\pi a t)^{-1/2} \int_0^\infty s^k e^{M\theta - (\tau-\theta)^2/(4at)} d\theta \le (k/(Me))^k e^{4M^2 at + 2M\tau}. \tag{9.2.7}$$

Indeed, the function $\theta^k e^{-M\theta}$ attains its maximum at $\theta = k/M$, hence $\theta^k e^{-M\theta} \le (k/(Me))^k$. So the integral in the left side of (9.2.7) is less than

$$(k/(Me))^k (4\pi a t)^{-1/2} \int_{\mathbb{R}} e^{2M\theta} e^{-(\tau-\theta)^2/(4at)} d\theta = (k/(Me))^k e^{4M^2 at + 2M\tau}$$

because due to the known integral formula for the Cauchy problem the both sides solve the same Cauchy problem for the heat equation $\partial_t v - a\partial_\tau^2 v = 0$.

Using the formula (9.2.4) and the first condition (9.2.6) we yield

$$|v(\tau,t)| \le (\pi t)^{-1/2}\int_0^\infty \theta^2 M^2 e^{M\theta-(\tau-\theta)^2/(4t)}d\theta \le 2M^2(2/(Me))^2 e^{4M^2t+2M\tau}$$

where we used the inequality (9.2.7) with $a=1$, $k=2$. So we have the first bound of Lemma 9.2.2. The second bound follows similarly when we use the integral formula (9.2.4) for $\partial_\tau v$ (with difference of exponents instead of the sum and with $\partial_\theta u(\theta)$ intstead of $u^*(\theta)$), second condition (9.2.5), the inequality (9.2.7) with $a=1$ and $k=1,3$, as well as the elementary inequality $2/e+(3/e)^3<2.1$. □

From Theorem 3.3.10 and from Lemma 9.2.2 we have

(9.2.8) $$|v| \le CC_1(M)\varepsilon^\kappa \text{ on } \Omega_{T,T^*}=\{0<\tau<T^*, T/4<t<3T/4\}$$

where $C_1(M)=e^{4M(MT+T^*)}$, $\kappa\in(0,1)$ and depends only on $T,T^*$. Now we will obtain the bound of Theorem 9.2.1 by using stability estimates of the analytic continuation of $v(,t)$ onto $(0,3T/4)$.

Our first claim is that for all $\tau\in(0,T^*)$ the function $v(\tau,t)$ has the complex-analytic continuation onto the sector $S=\{t=t_1+it_2: |t|<\sigma t_1\}$ of $\mathbb{C}$ and moreover

(9.2.9) $$|v| \le C_2(M)e^{\sigma M^2|t|}$$

where $C_2(M)=0.55\sqrt{\sigma}e^{2M\tau}$. This complex-analytic continuation for $0<t_2$ is given by the formula (9.2.4). Using in addition the bound $|e^{-(\tau-\theta)^2/(4t)}|\le e^{-(\tau-\theta)^2/(4\sigma|t|)}$ for $t\in S$ and conditions (9.2.6) we yield

$$|v(\tau,t)| \le M^2\sqrt{\sigma}(\sigma\pi|t|)^{-1/2}\int_0^\infty \theta^2 e^{M\theta-(\tau-\theta)^2/(4\sigma|t|)}d\theta$$

$$\sqrt{\sigma}M^2(2/(Me))^2 e^{4\sigma M^2|t|+2M\tau}$$

by Lemma 9.2.2 with $k=2$, $a=\sigma$. Using that $(2/e)^2<0.55$ we complete the proof of the bound (9.2.9).

The second claim is the following stability estimate for analytic continuation:

(9.2.10) $$|v(\tau,t)| \le CC_3(M)\varepsilon^{t^{\pi/(2\beta)}/C}$$

where $C_3(M)=\sqrt{\sigma}e^{\sigma^2M^2T+4MT^*}$, $\beta=cos^{-1}\sigma^{-1}$.

To prove (9.2.10) we introduce the function $V(\tau)=v(\tau)e^{-(\sigma^2M^2+\delta_1)t}$ with a positive parameter $\delta_1$. This function is complex-analytic in $S$ and from the bound (9.2.9) it follows that $|V(\tau,t)|\le C_2(M)e^{-\delta_1/\sigma|t|}$ when $t\in S$. Hence $log|V(\tau,)|$ is a subharmonic function on $S$ which tends to zero at infinity.

Let $\mu(t,I)$ be the harmonic measure of the interval $[T/4,3T/4]$ in $S$ with respect to $t$ which is defined in section 3.3. As in section 6.3 or in the proof of Theorem 9.4.3, by using the conformal mapping $z=t^{\pi/(2\beta)}$, $\beta=cos^{-1}\sigma^{-1}$ and the maximum principles one can show that $t^{\pi/(2\beta)}<C(\sigma)\mu(t,I)$ when $0<t<T$.

Since the subharmonic function $log|V| \leq logC_2(M)$ on $\partial S$ and due to (9.2.8)

$$log|V(,t)| \leq log(Ce^{4M(2Mt+T^*)}\varepsilon^\kappa e^{-\sigma^2 M^2 t}) \leq log(Ce^{4MT^*}\varepsilon^\kappa)$$

on $I$ when $3 < \sigma$, from maximum principles we derive that

$$log|V(,t)| \leq (1-\mu(t))logC_2(M) + \mu(t)(log(Ce^{4MT^*}) + \kappa log\varepsilon)$$

provided $t \in S$. Taking exponents of the both sides, replacing $\mu$ in the first term by 0, using the lower bound on $\mu$ and letting $\delta_1 \to 0$ we complete the proof of (9.2.10).

To conclude the proof of Theorem 9.2.1 we will utilize the Mean Value Theorem to get $v(\tau,0) = v(\tau,t) - \partial_t v(\tau,\theta(\tau,t))t$ for some $\theta \in (0,t)$. Using Lemma 9.2.2 and (9.2.10) we obtain

$$|v(\tau,0)| \leq CC_4(\sigma,M)(\varepsilon^{t^{\pi/(2\beta)}/C} + Mt).$$

Given $\delta > 0$ one can choose large $\sigma$ (hence $\beta$ close to $\pi/2$) so that $(2-\delta)\beta/\pi = 1-\delta$. Letting $t = (-log\varepsilon)^{-\beta/\pi(2-\delta)}$ in the bound for $v(\tau,0)$ we conclude that

$$|v(\tau,0)| \leq CC_4(\delta,M)(e^{-(-log\varepsilon)^{\delta/2}/C} + M(-log\varepsilon)^{-1+\delta}).$$

Finally, since $e^{-w^\delta/C} \leq C(\delta)w^{-1}$ we complete the proof of Theorem 9.2.1.

Returning to inverse problems, we first give a complete solution of the uniqueness question in the one-dimensional case for some special lateral boundary data.

**Theorem 9.2.3.** *Let* $\Gamma = \partial\Omega$, *where* $\Omega = (0,1)$ *in* $\mathbb{R}$. *Let* $g_0(t,j)$, $j = 0,1$, *be the transform* (9.2.1) *of a function* $g_0^*(\tau,j) \in C^k([0,\infty))$ *whose absolute value is bounded by* $C\exp(C\tau)$ *with some* $C$ *and that satisfies the condition* $g_0^{*(k-1)}(0) \neq 0$.

*Then (i) the coefficient* $c = c(x) \in L_\infty(\Omega)$ *of equation* (9.0.1) *with* $a_0 = a = 1, b = 0$ *or (ii) the coefficient* $a = a(x) \in C^2(\overline{\Omega})$ *of the same equation with* $a_0 = 1, b = 0, c = 0$ *is uniquely determined by the Neumann data* (9.0.6) *of the parabolic problem* (9.0.1)–(9.0.3) *with* $f = 0$, $u_0 = 0$, *and* $g = g(,j)$ *at* $x = j$.

PROOF. Case (i). Let us consider the hyperbolic problem

$$\partial_\tau^2 u^* - \partial_x^2 u^* + c(x)u^* = 0 \text{ in } \Omega \times (0,T^*),$$

$$u^* = \partial_\tau u^* = 0 \text{ on } \Omega \times \{0\},$$

$$u^* = g_0^* \text{ on } (0,T^*) \times \partial\Omega. \tag{9.2.11}$$

Using (9.2.3) and the remarks after it concerning the transformation of $\partial_\tau^2 u^*$, we conclude that $u$ obtained via (9.2.1) solves the parabolic equation (9.0.1), (where $a_0 = 1, a = 1, b = 0$). The additional data

$$\partial_x u^* = g_1^* \text{ on } \quad \partial\Omega \times (0,T^*) \tag{9.2.12}$$

necessary to find the coefficient $c(x)$ in (9.2.11) can be obtained by inverting the relation (9.2.1), where $u$ is replaced by $g_1$. As follows from Theorem 9.2.1, the correspondence between $g_1^*$ and $g_1$ is unique, so we can uniquely determine $g_1^*$.

According to known results about one-dimensional inverse hyperbolic problems (Corollary 8.1.2), a solution $c(x)$ to the inverse problem (9.2.11), (9.2.12) is unique (and stable) on $(0, \frac{1}{2})$ if we choose $T^* = 1$. With this choice of $T^*$ we can use Corollary 8.1.2, because due to finite speed of propagation, $g_1^*$ on $(0, T^*)$ coincides with the Neumann data for the hyperbolic problem when $\Omega = (0, \infty)$. To prove uniqueness of $c$ on $(\frac{1}{2}, 1)$, one can similarly use the data $g_1^*$ at $x = 1$.

Part (ii) follows from Corollary 8.1.7, after a similar reduction to the inverse hyperbolic problem with variable speed of propagation.

The proof is complete. □

As an explicit example of the Dirichlet data satisfying the conditions of Theorem 9.2.3 we can use $g_0(t) = 1$. The corresponding function $g_0^*(\tau) = 1/2\tau^2$. This can be seen from the parabolic intial value problem (9.2.3) with $v(t, \tau) = t - 1/2\tau^2$.

The unstable part of the reconstruction procedure described in this proof is the step $g_1 \to g_1^*$. On the other hand, this logarithmic instability is isolated and is due to the solution of the simplest, standard ill-posed lateral Cauchy problem for the one-dimensional heat equation (9.2.3), which is relatively well understood.

In the multidimensional case the best uniqueness results are available in the case of nonzero initial data $u$. Let $\mathcal{P}$ be a half-space in $\mathbb{R}^n$, $e$ be the exterior unit normal to $\partial\mathcal{P}$, $\gamma$ equal to $\partial\Omega \cap \mathcal{P}$, and $x_0$ a point of $\mathcal{P}$ such that $x^0 \cdot e \geq x \cdot e$ when $x \in \gamma$.

**Theorem 9.2.4.** *Let* $a = 1, b = 0, c = 0, f = 0,$ *and* $g_0^*, u_0 \in C^l(\overline{\Omega}), \partial\Omega \in C^l$ *where* $(n + 7)/2 \leq l$. *Let us assume that*

$$0 < \varepsilon_0 < \Delta u_0 \quad on\ \Omega_0 = \Omega \cap \mathcal{P}. \tag{9.2.13}$$

*Then the coefficient* $a_0 = a_0(x) \in C^l(\overline{\Omega_0})$ *of the parabolic equation (9.0.1) with the given initial data (9.0.3) and satisfying condition*

$$\nabla a_0 \cdot e \leq 0, 0 \leq a_0 + \frac{1}{2}\nabla a_0 \cdot (x - x_0)\ on\ \Omega_0 \tag{9.2.14}$$

*is uniquely determined on* $\Omega_0$ *by the Cauchy data*

$$u = g_0, \partial_\nu u = g_1\ on\ \gamma \times (0, T).$$

PROOF. We will make use of the transform (9.2.1) again, this time reducing our parabolic problem to the following hyperbolic one:

$$a_0 \partial_t^2 u^* - \Delta u^* = 0 \qquad \text{on } \Omega \times (0, T^*),$$
$$u^* = u_0, \partial_\tau u^* = 0 \text{ on } \Omega \times \{0\},$$
$$u^* = g_0^* \text{ on } \partial\Omega \times (0, T^*).$$

By uniqueness of inversion $g_1 \to g_1^*$ due to Theorem 9.2.1, we conclude that the data of the inverse parabolic problem uniquely identify the data of the inverse hyperbolic problem for any $T^* > 0$. Now uniqueness of $a_0$ follows from Corollary 8.2.3.

The proof is complete. □

**Exercise 9.2.5.** Show that in the situation of Theorem 9.2.4 the coefficient $c = c(x) \in L_\infty(\Omega)$ of the equation $\partial_t u - \Delta u + cu = 0$ is uniquely determined by the additional lateral Neumann data (9.0.6), provided that condition (9.2.14) is replaced by the condition $0 < \varepsilon_0 < u_0$ on $\Omega_0$.

A disadvantage of Theorem 9.2.4 and of similar results is the condition that the initial condition is not zero. This condition is not satisfies in many applications when the physical field is intiated from the lateral boundary. Another seemingly excessive restriction of Theorem 9.2.4 is the condition (9.2.14) which guarantees absence of trapped rays/validity of appropriate Carleman estimates in the associated hyperbolic problem.

One of few available theoretical results in case of zero initial data and few boundary measurements requires simulteneous overdetermination at $\partial\Omega$ and inside $\Omega$ at some fixed moment of time $\theta \in (0, T)$.

**Theorem 9.2.6.** *Let us assume that $a_0, a, b, c \in C^1(\overline{Q})$ and $\alpha, \partial_t\alpha \in C(\overline{Q})$. Let $\gamma$ be any open subset of $\partial\Omega$ and $\Gamma = \gamma \times (0, T)$. Let $\theta \in (0, T)$ and*

$$\varepsilon_0 < \alpha \text{ on } \Omega \times \{\theta\} \tag{9.2.15}$$

*for some positive number $\varepsilon_0$.*

*Then any pair $(u, F) \in H_{2,1;2}(\Omega) \times L_2(\Omega)$ satisfying the equation (9.0.1) with $f = \alpha F$, $\partial_t F = 0$ in $Q$ is uniquely determined by the lateral Cauchy data*

$$u = g_0,\ \ \partial_\nu u = g_1 \text{ on } \gamma \times (0, T)$$

*and the intermediate time data*

$$u(, \theta) = u_\theta \text{ on } \Omega.$$

This result follows from Theorem 8.2.2 because as shown in the proof of Theorem 3.3.10 for any subdomain $\Omega_0$ of $\Omega$ which is a diffeomorphic image of a halfsphere, so that $\gamma_0 = \partial\Omega_0 \cup \partial\Omega$ is in $\gamma$, there is a Carleman estimate (3.2.3) for the equation (9.0.1) with the weight function $\varphi$ with $\varphi > 0$ on $\gamma_0 \times (\theta - \varepsilon, \theta + \varepsilon)$ and $\varphi < 0$ on $Q \backslash (\Omega_0 \times (\theta - \varepsilon, \theta + \varepsilon))$. We would like to emphasize locality of this uniqueness statement: indeed we do not assume that the lateral boundary data are known outside $\Gamma$, so $\Omega$ can be an arbitrary subdomain of a larger domain. We only need condition (9.2.15).

By using some new Carleman estimates and assuming additional regularity of the coefficients of the parabolic boundary value problem Imanuvilov and Yamamoto [IY1] showed that under the additional assumption that $u$ satisfies a lateral boundary condition $u = 0$ or $\partial_{\nu(a)}u + b_0 u = 0$ on $\partial\Omega \times (0, T)$ one has Lipschitz stability estimate

$$\|F\|_2(\Omega) \leq C(\|u_\theta\|_{(2)}(\Omega) + \sum \|w\,\partial_t^l \partial_j^m\|_2(\Gamma))$$

where the sum is over $j = 1, \ldots, n$, $l, m = 0, 1$, $w$ is some nonnegative $C^2(Q)$-function which goes to infinity as $t$ goes to 0 or $T$. In this stability estimate $C$ depends on the coefficients of the parabolic initial boundary value problem, on $\Omega, T, \gamma$, $\|\alpha\|_\infty(Q) + \|\partial\alpha\|_\infty(Q)$ and on $\varepsilon_0$ from condition (9.2.15). This a best possible estimate showing a possibility for a very efficient numerical solution of this inverse problem.

## 9.3 The inverse problem of option pricing

As an important example of inverse problem with single measurements for parabolic equations we will consider determination of so-called volatility coefficient $\sigma$ of a parabolic equation for option prices discovered by Black and Scholes in 1973. This discovery revolutionized financial markets in 1990-s. In 1997 Merton and Scholes were awarded the Nobel prize in economics.

For any stock price, $0 < s < \infty$, and time $t, 0 < t < T$, the price $u$ for an option expiring at time $T$ satisfies the following Black-Scholes partial differential equation

(9.3.1) $$\partial u/\partial t + 1/2 s^2\sigma^2(s,t)\partial^2 u/\partial s^2 + s\mu\partial u/\partial s - ru = 0.$$

Here, $\sigma(s,t)$ is the volatility coefficient that satisfies $0 < m < \sigma(s,t) < M < \infty$ and is assumed to belong to the Hölder space $C^\lambda(\bar{\omega})$, $0 < \lambda < 1$, on some interval $\omega$ and outside this interval, and $\mu$ and $r$ are, respectively, the risk-neutral drift and the risk-free interest rate assumed to be constants. The backward in time parabolic equation (9.3.1) is augmented by the final condition specified by the payoff of the call option with the strike price $K$

(9.3.2) $$u(s,T) = (s-K)^+ = max(0, s-K),\ \ 0 < s.$$

By using the logarithmic substitution $y = logs$ and Theorem 9.1 one can show that there is a unique solution $u$ to (9.3.1), (9.3.2) which belongs to $C^1((0,\infty)\times(0,T])$ and to $C((0,\infty)\times[0,T])$ and satisfies the bound $|u(s,t)| < C(s+1)$.

All coefficients of the equation (9.3.1) except of $\sigma$ are known. Volatility coefficient is a fundamental characteristic of options market, so it is highly desirable to know it. The inverse problem of option pricing seeks for $\sigma$ given

(9.3.3) $$u(s^*, t^*; K, T) = u^*(K),\ \ K \in \omega^*.$$

Here $s^*$ is market price of the stock at time $t^*$, and $u^*(K)$ denote market price of options with different strikes $K$ for a given expiry $T$. The additional data (9.3.3) are available from current trading. One can find them on Internet. We will attempt to recover volatility in the same interval $\omega^*$ containing $s^*$.

To obtain our results we will use that the option premium $u(.,.;K,T)$ satisfies the equation dual to the Black-Scholes equation (9.3.1):

(9.3.4) $$\partial u/\partial T - 1/2K^2\sigma^2(K,T)\partial^2 u/(\partial K^2) + \mu K\partial u/(\partial K) + (r-\mu)u = 0.$$

The equation (9.3.4) was found by Dupire in 1994 and rigorously justified, for example, in [BI].

The logarithmic substitution

$$y = ln(K/s^*),\ \ \tau = T-t,\ \ U(y,\tau) = u(;K,T),\ \ a(y,\tau) = \sigma(s^*e^y, T-\tau),$$
(9.3.5)

transforms the dual equation (9.3.4) and the additional (market) data into the following inverse parabolic problem

$$\partial U/\partial\tau = 1/2a^2(y,\tau)\partial^2 U/\partial y^2 - (1/2a^2(y,\tau)+\mu)\partial U/\partial y$$
$$+(\mu - r)U,\ y\in\mathbb{R},\ 0<\tau<T,$$

(9.3.7) $$U(y,0) = s^*(1-e^y)^+,\ y\in\mathbb{R}$$

with the final observation

(9.3.8) $$U(y,\tau^*) = U^*(y),\ y\in\omega.$$

where $\omega$ is the interval $\omega^*$ in $y$-variables.

We list available theoretical results. $\sigma(s,t)$ can be found from the data $U^*(K,T)$ due to the Dupire's equation (9.3.4). In many important cases temporal data are not available or sparse. In any case these data are not available for future, when knowledge of volatility is most important for useful predictions. This is why we think it is reasonable to look for $\sigma = \sigma(s)$ and that what we will do in the remaining part of section 9.3.

**Theorem 9.3.1.** *Let $U_1$ and $U_2$ be two solutions to the initial value problem with $a = a_1$ and $a = a_2$ and let $U_1^*, U_2^*$ be the corresponding final data. Let $\omega_0$ be a non-void open subinterval of $\omega$.*

*If $U_1^*(y) = U_2^*(y)$ for $y\in\omega$ and $a_1(y) = a_2(y)$ for $y\in\omega_0$, then $a_1(y) = a_2(y)$ when $y\in\omega$.*

*If, in addition, $a_1(y) = a_2(y)$ when $y\in\omega\cup(\mathbf{R}\backslash\omega)$ and if $\omega$ is bounded, then there is a constant $C$ depending only on $|a_1|_1(\omega)$, $|a_2|_2(\omega)$, $\omega$, $\omega_0$, $\tau^*$, $\lambda$ such that*

$$|a_2 - a_1|_\lambda(\omega) \le C|U_2^* - U_1^*|_{2+\lambda}(\omega)$$

PROOF. To show uniqueness we subtract two equations for $U_2$ and $U_1$ to get

$$\partial U/\partial\tau = 1/2a_2^2(y)\partial^2 U/\partial y^2 - (1/2a_2^2(y)+\mu)\partial U/\partial y + (\mu-r)U + \alpha_1(y,\tau)f(y)$$

where

$$U = U_2 - U_1,\ \alpha_1 = \partial^2 U_1/\partial y^2 - \partial U_1/\partial y,\ f(y) = 1/2(a_2^2(y) - a_1^2(y)).$$

Besides, $U(y,0) = 0$. By Theorem 9.5 solutions of an initial value parabolic problem with time independent coefficients are time analytic. Since $U(,\tau^*) = 0$, $f = 0$ on $\omega_0$ we conclude from the differential equation that $\partial U/(\partial\tau)(,\tau^*) = 0$ on $\omega_0$. Repeating this argument we conclude that all $\tau$-derivatives of $U$ are zero on $\omega_0\times\{\tau^*\}$. By analyticity $U = 0$ on $\omega_0\times(0,\tau^*)$. By using (9.3.7) the function $\alpha_1$ satisfies (in distributional/generalized sense) the initial value problem

$$\partial_\tau\alpha_1 = 1/2(\partial_y^2 - \partial_y)(a_1^2(y)\alpha_1) - 1/2\mu\partial_y\alpha_1 + (\mu-r)\alpha_1,\ y\in\mathbb{R},\ 0<\tau<T$$

$$\alpha_1(y,0) = \delta(y)$$

where $\delta$ is the Dirac delta-function. So $\alpha_1$ is the Green's function for this Cauchy problem and hence $\alpha(y,\tau) > 0$. By Theorem 9.2.6 $f = 0$ on $\omega$.

Stability estimate follows from uniqueness and Theorems 9.1, 9.1.1 by compactness-uniqueness arguments. □

The assumption that $a(y)$ is known on a subinterval of $\omega$ is probably not nesessary. Moreover it prevents from existence results since it severely overdetermines the inverse problem. Masahiro Yamamoto observed that for $a \in C^\infty(\mathbb{R})$ this assumption can be removed.

A feature of the inverse options pricing problem is localization around the underlying price $s^*$ resulting from singularity of the final data. Thus local results make sense.

**Theorem 9.3.2.** *Let $|a|_\lambda(\omega) < M$ and $a^2(y) = \sigma_0^2$ (given constant) on $\mathbf{R} \setminus \omega$.*

*Then there is $\varepsilon > 0$ (depending only on $s^*, \tau^*, \sigma_0$) and $M$ such that under the condition $\omega \subset (-\varepsilon, \varepsilon)$ a solution $a(y)$ to the inverse problem (9.3.1)-(9.3.3) is unique.*

A proof follows from standard contraction arguments augmented by careful study of singularities of solution and it is based on the study of the linearized inverse problem given below. For details of proofs of Theorems 9.3.1, 9.3.2 we refer to the review paper [BI].

In the remaining part of section 9.3 we will assume that $1/2\sigma^2(s) = 1/2\sigma_0^2 + f_*(s)$ where $f_*$ is a small $C(\bar{\omega})$-perturbation of constant $\sigma_0^2$ and $f_* = 0$ outside $\omega_*$.

As in section 4.5 one can show that $U = V_0 + V + v$. Here $V_0$ solves (9.3.10) with $a = \sigma_0$ and $v$ is quadratically small with respect to $f_*$, while the principal linear term $V$ satisfies the equations

$$\partial V/(\partial\tau) - 1/2\sigma_0^2\partial^2 V/(\partial y^2) + (\sigma_0^2/2 + \mu)\partial V/(\partial y) + (r - \mu)V = \alpha_0 f,$$

$$V(y, 0) = 0, \quad y \in \mathbb{R},\ V(y, \tau^*) = V^*(y), \quad y \in \omega,$$

where

$$\alpha_0(y, \tau) = s^* 1/\sigma_0(4\pi a\tau)^{-1/2} e^{-y^2/(2\tau\sigma_0^2)} + cy + d\tau,$$

$$c = 1/2 + \mu/\sigma_0^2, \quad d = -1/(2\sigma_0^2)(\sigma_0^2/2 + \mu)^2 + \mu - r$$

and $V^*$ is the principal linear part of $U^*$.

The new substitution $V = e^{cy+d\tau} W$ simplifies these equations to

$$\partial W/(\partial\tau) - 1/2\sigma_0^2\partial^2 W/(\partial y^2) = \alpha f, \quad 0 < \tau < \tau^*, \quad y \in \mathbb{R},$$

$$W(y, 0) = 0, \quad y \in \mathbb{R},\ W(y, \tau^*) = W^*(y), \quad y \in \omega, \tag{9.3.9}$$

with

$$\alpha(\tau, y) = s^*/(\sqrt{2\pi\tau}\sigma_0)e^{-y^2/(2\tau\sigma_0^2)},\ W^*(y) = e^{-cy-d\tau^*}V^*(y).$$

In the remaining part of this section we will assume that $f = 0$ outside $\omega$. From numerical experiments ( in particular, in [BIV]) we can see that values of $f$ outside $\omega$ are not essential, due to a very fast decay of the Gaussian kernel $\alpha$ in $s$.

Let us denote by $Af$ the solution to (9.3.9) on $\omega : Af(y) = W(y, \tau^*), \quad y \in \omega$.

**Lemma 9.3.3.** *We have*

$$Af(x) = \int_\omega B(x, y; \tau^*) f(y) dy, \quad x \in \omega,$$

*where*

$$B(x, y; \tau^*) = s^*/(\sigma_0^2 \sqrt{\pi}) \int_{(|x-y|+|y|)/(\sigma_0 \sqrt{2\tau^*})} e^{-\tau^2} d\tau.$$

*Moreover, the linearized inverse problem implies the following Fredholm integral equation*

$$f(x) - 1/(2\tau^* \sigma_0^2) \int_\omega e^{-((|x-y|+|y|)^2 - |x|^2)/(2\tau^* \sigma_0^2)} (|x-y| + |y|) f(y) dy$$

$$= -\sqrt{\pi \tau^*}/\sqrt{2} s^* \sigma_0^3 e^{(|x|^2)/(2\tau^* \sigma_0^2)} \partial^2/(\partial x^2) W(x, \tau^*), \; x \in \omega \tag{9.3.10}$$

PROOF. The well-known representation of the solution to the Cauchy problem (9.3.9) for the heat equation yields

$$W(x, \tau) = \int_{\mathbb{R}} B(x, y; \tau) f(y) dy,$$

$$B(x, y; \tau) = \int_0^\tau 1/(\sqrt{2\pi(\tau-\theta)}\sigma_0) e^{-|x-y|^2/(2\sigma_0^2(\tau-\theta))} s^*/(\sqrt{2\pi\theta}\sigma_0) e^{-|y|^2/(2\sigma_0^2\theta)} d\theta.$$

We will simplify $B(x, y; \tau)$ by using the Laplace transform $\Phi(p) = \mathcal{L}(\phi)(p)$ of $\phi(\tau)$ with respect to $\tau$. Since the Laplace transform of the convolution is the product of Laplace transforms of convoluted functions, we have

$$\mathcal{L}B(x, y; )(p) = s^*/(2\pi\sigma_0^2) \mathcal{L}(\tau^{-1/2} e^{-|x-y|^2/(2\sigma_0^2\tau)}) \mathcal{L}(\tau^{-1/2} e^{-|y|^2/(2\sigma_0^2\tau)})$$

$$= s^*/(2\pi\sigma_0^2) \sqrt{\pi/p} e^{-\sqrt{2}|x-y|/\sigma_0} \sqrt{p} \sqrt{\pi/p} e^{-\sqrt{2}|y|/\sigma_0 \sqrt{p}}$$
$$= s^*/(2\sigma_0^2) 1/p e^{-\sqrt{2}(|x-y|+|y|)/\sigma_0 \sqrt{p}},$$

where we used the formula for the Laplace transform of $\tau^{-1/2} e^{-\beta/\tau}$. Applying the formula for the inverse Laplace transform of the function $1/pe^{-\alpha\sqrt{p}}$ we arrive at the conclusion of Lemma 9.3.3.

A proof of the second statement can be obtained by differentiating the equation $Af = W(, \tau^*)$ on $\omega$. □

**Theorem 9.3.4.** *Let $\omega = (-b, b)$. Let $\theta_0$ be the root of the equation $2\theta - e^{-4\theta} = 3$. If*

$$b^2/(\tau^* \sigma_0^2) < \theta_0, \tag{9.3.11}$$

*then a solution $f \in L_\infty(\omega)$ to the integral equation (9.3.11) and hence to the linearized inverse option pricing problem (9.3.9) is unique.*

PROOF. Due to Lemma 9.3.3 to prove Theorem 9.3.4 it suffices to show uniqueness of solution $f$ of (9.3.10). To do it we observe that

$$\int_\omega e^{-(|x-y|+|y|)^2/(2\tau^*\sigma_0^2)}(|x-y|+|y|)dy$$

$$= e^{-x^2/(2\tau^*\sigma_0^2)}(\tau^*\sigma_0^2 + x^2) - (\tau^*\sigma_0^2)/2(e^{-(2b-x)^2/(2\tau^*\sigma_0^2)} + e^{-(2b+x)^2/(2\tau^*\sigma_0^2)}).$$

(9.3.12)

Returning to uniqueness of $f$ we assume that $f$ is not zero. We can assume that $\|f\|_\infty(\omega) = f(x_0) > 0$ at some $x_0 \in [-b, b]$. From (9.3.10) at $x = x_0$ (with zero right side) we have

$$\|f\|_\infty \le 1/(2\tau^*\sigma_0^2)\int_\omega e^{-((|x_0-y|+|y|)^2-x_0^2)/(2\tau^*\sigma_0^2)}(|x_0-y|+|y|)\|f\|_\infty dy$$

$$= \|f\|_\infty((\tau^*\sigma_0^2 + x_0^2)/(2\tau^*\sigma_0^2) - 1/4(e^{-((2b-x_0)^2-x_0^2)/(2\tau^*\sigma_0^2)} + e^{-((2b+x_0)^2-x_0^2)/(2\tau^*\sigma_0^2)}))$$

if we use (9.3.12).

One can show that (9.3.11) implies

$$g(x) = x^2/(\tau^*\sigma_0^2) - 1/2(e^{(2b(x-b))/(\tau^*\sigma_0^2)} + e^{-(2b(x+b))/(\tau^*\sigma_0^2)}) < 1, \quad -b \le x \le b.$$

Then the previous inequality yields

$$\|f\|_\infty < \|f\|_\infty(1/2 + 1/2) = \|f\|_\infty,$$

and hence $\|f\|_\infty(\omega) = 0$. □

By Lemma 9.3.3 the linearized inverse option pricing problem implies the integral equation

(9.3.13) $$Af(x) = W^*(x),\ x \in \omega = (-b, b),$$

where $W^*$ is the function defined after (9.3.9). Lemma 9.3.3 and Theorem 9.3.4 guarantee uniqueness a solution $f \in C(\bar\omega)$ to this equation under the condition (9.3.11). It is not known whether this condition is necessary for uniqueness in the linearized inverse problem.

One can show [BIV] that the range of $A$ has the codimension 2 in $C^{2+\lambda}(\bar\omega)$. At present we do not know an exact description of the range of $A$.

The integral equation (9.3.13) with the data $W^*(x)$ equal to the difference of the final states $e^{-cy-d\tau^*}(U^*(y) - U_0^*(y))$ where $U$ solves the parabolic equation (9.3.7) with $a^2(y) = \sigma_0^2 + 2f(y)$ and $U_0$ solves the unperturbed equation (9.3.7) was solved numerically in [BIV] with good results indicating that the condition (9.3.11) of Theorem 9.3.4 can be essential. Moreover, numerics worked very well when the (uniform) deviation of $\sigma$ from $\sigma_0 = 1$ did not exceed 0.3.

## 9.4 Lateral overdetermination: many measurements

This section is devoted to identification problems when one is given the Neumann data (9.0.6) for all (regular) Dirichlet data (9.0.3). In other words, we know the so-called lateral Dirichlet-to-Neumann map $\Lambda_l : g_0 \to g_1$. We will assume that unknown coefficients are regular; in particular, the principal coefficients $a$ must be at least $C^1$. Discontinuous principal coefficients are to be considered in section 9.5.

**Theorem 9.4.1.** *Let* $a_0 = 1, b = 0$, *and let* $a$ *be a scalar matrix. Let* $f = 0$ *and* $u_0 = 0$. *Let* $\Gamma = \partial\Omega$.

*Then the lateral Dirichlet-to-Neumann map* $\Lambda_l$ *uniquely determines* $a \in H_{2,\infty}(\Omega)$ *and* $c \in L_\infty(\Omega)$.

PROOF. We will make use of stabilization of solutions of parabolic problems when $t \to \infty$, reducing our inverse parabolic problem to inverse elliptic problems with parameter, where one can use results from Chapter 5.

The substitution

$$u = ve^{\lambda t} \tag{9.4.1}$$

transforms equation (9.0.1) into the equation

$$\partial_t v - \operatorname{div}(a\nabla v) + (c + \lambda)v = 0 \text{ on } Q. \tag{9.4.2}$$

Choosing $\lambda$ large, we can achieve that $0 \le c + \lambda$. It is clear that the lateral Dirichlet-to-Neumann map for $v$ is given as well.

Let $g_0 = g^0\phi$, where $g^0$ is any function in $C^2(\overline{\Omega})$ and $\phi(t) \in C^\infty(\mathbb{R})$ satisfies the conditions $\phi(t) = 0$ on $(-\infty, T/4)$ and $\phi(t) = e^{\lambda t}$ on $(T/2, +\infty)$. Since the coefficients of equation (9.4.2) and the lateral boundary condition do not depend on $t > T/2$, the solution $u(t, x)$ is analytic with respect to $t > T/2$ as well as $\partial_\nu u(t, x)$, $x \in \partial\Omega$.

Since our equation (9.4.2) satisfies all the conditions of the maximum principle, and its coefficients and the lateral boundary condition do not depend on $t > T/2$, Theorem 9.6 guarantees that

$$\|v(\ , t) - v^0\|_\infty(\Omega) \to 0 \text{ as } t \to \infty, \tag{9.4.3}$$

where $v^0$ is a solution to the (stationary) Dirichlet problem

$$-\operatorname{div}(a\nabla v^0) + (c + \lambda)v^0 = 0 \text{ in } \Omega,\ v^0 = g^0 \text{ on } \partial\Omega. \tag{9.4.4}$$

The maximum principle for parabolic equations yields $\|v\|_\infty(\Omega \times \mathbb{R}^+) \le C$, so the local $L_p$-estimates for parabolic boundary value problems (Theorem 9.1) give $\|v\|_{2,1;p}(\Omega \times (s - 1, s + 1)) \le C$. Differentiating the equation with respect to $t$ and again using local estimates, we derive from the previous bound that $\|\partial_t v\|_{2,1;p}(\Omega \times (s, s + 1)) \le C$. So trace theorems give $\|v(, t)\|_{(2)}(\Omega) \le C$. From this estimate and from the estimate (9.4.3), by interpolation theorems it follows that

$$\|v(\,,t)-v^0\|_{(1)}(\Omega)\to 0 \text{ as } t\to\infty. \tag{9.4.5}$$

Again by trace theorems, we obtain that $\partial_\nu v(,t)\to\partial_\nu v^0$ in $H_{(-1/2)}(\partial\Omega)$ as $t\to\infty$. Since $a\partial_\nu v(,t)$ on $\partial\Omega$ is given, we conclude that $a\partial_\nu v^0$ on $\partial\Omega$ is given as well. Thus, we are given the Dirichlet-to-Neumann map for the elliptic equation (9.4.4) (for all $\lambda\geq -c$). As in Theorem 5.1.1, $a$ and $\nabla a$ on $\partial\Omega$ are uniquely determined.

The known Riccati substitution $v^0=a^{-1/2}w$ as in Section 5.2 transforms equation (9.4.4) into the Schrödinger equation

$$-\Delta w+a^{-1/2}(\Delta a^{1/2}+a^{-1/2}(c+\lambda))w=0 \quad \text{in } \Omega. \tag{9.4.6}$$

By Theorem 5.3.1 ($3\leq n$) and Corollary 5.5.2 ($n=2$), the coefficient $c^*$ of $w$ in equation (9.4.6) (for any $\lambda$) is uniquely determined by its Dirichlet-to-Neumann map. Hence $a^{-1/2}$ is uniquely determined as the coefficient of $\lambda$ in $c^*$, and therefore $c$ is uniquely determined as well.

The proof is complete. □

With the appropriate stabilization theory for equations with $L_\infty(\Omega)$ coefficient $a$ the method of the proof of Theorem 9.4.1 and Theorem 5.4.1 will imply uniqueness of $a$ provided $b=0, c=0$.

The method of proof of this theorem and Theorem 5.5.1 can be used to solve the following exercise.

**Exercise 9.4.2.** Let $a=1$ and $3\leq n$. Show that $\Lambda_l$ uniquely determines the coefficients $a_0(x)\in L_\infty(\Omega)$, curl $b(b\in H_{2,\infty}(\Omega))$ and $4c+b\cdot b-2\operatorname{div} b$, $c\in L_\infty(\Omega)$, in equation (9.0.1) with $f=0$, provided that $\|b\|_{1,\infty}(\Omega)$ is small.

The stabilization technique also enables one to obtain stability estimates. Let us consider two lateral Dirichlet-to-Neumann maps $\Lambda_{l1}$ and $\Lambda_{l2}$ corresponding to equation (9.0.1) with the coefficients $a_0=a_{01}$ and $a_{02}$, $a=1$, $b=0$, $c=c_1$ and $c_2$ that depend only on $x$ and $f=0$. Let $\varepsilon$ be the operator norm of $\Lambda_{l1}-\Lambda_{l2}$ (from $L_2((0,T);H_{(1/2)}(\partial\Omega))$ into $L_2((0,T);H_{(-1/2)}(\partial\Omega))$.

**Theorem 9.4.3.** *Assume that* $n=3$ *and*

$$\|a_{0j}\|_{\infty,1}(\Omega)+\|c_j\|_{\infty,1}(\Omega)<M. \tag{9.4.7}$$

*Then there are constants* $C,\delta\in(0,1)$ *such that*

$$\|a_{01}-a_{02}\|_\infty(\Omega)+\|c_1-c_2\|_\infty(\Omega)<C|\ln\varepsilon|^{-1/6}. \tag{9.4.8}$$

PROOF. Using the substitution (9.4.1), we can achieve that $0\leq c_j+\lambda a_{0j}$.

Let $g^0\in C^2(\partial\Omega)$, $\|g^0\|_{(1/2)}(\partial\Omega)<1$. We consider $g_0=g^0\phi$, where $\phi$ is as introduced in the proof of Theorem 9.3.1. Let $u_1,u_2$ be solutions to the initial boundary value problems (9.0.1)–(9.0.3) with the coefficients $a_{01},a_{02},\ldots$, with zero initial condition (9.0.2), and with lateral Dirichlet data $g_0$. By Theorem 9.5,

$u_j(x,t)$ has a complex-analytic extension onto the sector $S = \{t = t_1 + it_2 : |t_2| \leq (t_1 - T/2)\}$ of the complex $t$-plane. Moreover, the extension operator is bounded, so $\|u_j(,t)\|_{(1)}(\Omega) \leq Ce^{Ct_1}$ when $t \in S$, with $C$ possibly depending on $\Omega$, $\phi$, and $M$. By trace theorems,

$$\|c^{-Ct}\partial_\nu(u_1 - u_2)(\ ,t)\|_{(-1/2)}(\partial\Omega) \leq C \text{ when } t \in S. \tag{9.4.9}$$

Since the norm of a complex-analytic function with values in a Banach space is a subharmonic function in the plane, we conclude that the function $s(t) = \ln\|e^{-Ct}\partial_\nu(u_1 - u_2)\|_{(-1/2)}(\partial\Omega)$ is subharmonic on $S$. Let $\mu(t)$ be the harmonic measure of the interval $I_T = (7T/8, T)$ in $S$. We recall that $\mu$ is a harmonic function on $S\backslash\overline{I_T}$ that is continuous in $\overline{S\backslash I_T}$ and is equal to zero on $\partial S$, to 1 on $I_T$, and tends to zero when $t$ goes to infinity. By using conformal mappings onto a standard annulus, it is not difficult to show that $\mu$ exists. Observe that using the conformal mapping $\tau_1 = (t - T/2)^2$ of $S$ onto the half-plane $\{0 < \Re\tau_1\}$ and then the conformal mapping $\tau = 2/(1 + \tau_1)$, one concludes that $\mu$ (in $\tau$-variables) is harmonic in $B_1\backslash\gamma$, where $B_1$ is the disk $\{|\tau - 1| < 1\}$ and $\gamma$ is the subinterval of $(0,1)$ that is the image of $(7T/8, T)$ under the map $t \to \tau$. In addition, $\mu = 0$ on $\partial B_1$ and $\mu = 1$ on $\gamma$. Since $\mu$ achieves its minimum at $\tau = 0$, from the maximum principles for elliptic equations it follows that $\partial_1\mu(0) > 0$, so $\mu(\tau) > \tau/C$ when $0 < \tau$ for some constant $C$. Returning to $t$-variables, we conclude that $t^{-2}/C < \mu(t)$ when $T < t$.

We claim that

$$s(t) \leq C\mu(t)\ln\varepsilon + (1 - \mu(t))\ln C \text{ when } t \in S.$$

Indeed, when $t \in \partial S$, this follows from inequality (9.4.9). When $t \in I_T$, it follows from our definition of $\varepsilon$. Since the right side is harmonic and the left side is subharmonic in $S\backslash\overline{I_T}$, the maximum principles imply the inequality in $S$. Taking exponents of both parts of this inequality, we obtain

$$\|\partial_\nu(u_1 - u_2)\|_{(-1/2)}(\partial\Omega \leq Ce^{Ct}\varepsilon^{\mu(t)}C^{1-\mu(t)} \leq C^2e^{Ct}\varepsilon^{t^{-2/C}},\ T < t, \tag{9.4.10}$$

when we use one of the properties of $\mu$ and drop the $\mu$ in the exponent. On the other hand, by Theorem 9.6 we have exponential stabilization of the solution $u_j(,t)$ of our parabolic problem to the solution $u_j(,t)$ of our parabolic problem to the solution $u_{0j}$ to the steady-state elliptic problem

$$-\Delta u_{0j} + (c_j + \lambda a_{0j})u_{0j} = 0 \text{ in } \Omega,\ u_{0j} = g^0 \text{ on } \partial\Omega, \tag{9.4.11}$$

which gives $\|u_j(,t) - u_{0j}\|_\infty(\Omega) \leq Ce^{-\theta t}$, where positive $\theta$ depends only on $\Omega$ and the upper bounds on the coefficients of the operator. From this bound and from interior Schauder-type estimates (Theorem 9.1) for the differences $u_j - u_{0j}$ in the domains $\Omega \times (\Theta, \Theta + 1)$ we obtain the bounds of the $H_{(1)}(\Omega)$-norms of these differences. By using trace theorems in $\Omega$ we get

$$\|\partial_\nu(u_j(,t) - u_{0j})\|_{(-1/2)}(\partial\Omega) \leq Ce^{-\theta t}. \tag{9.4.12}$$

The triangle inequality for norms yields

$$\begin{aligned}&\|\partial_\nu(u_{01}-u_{02})\|_{(-1/2)}(\partial\Omega)\\&\quad\le \|\partial_\nu(u_1(\ ,t)-u_2(,t))\|_{(-1/2)}(\partial\Omega)+\|\partial_\nu(u_1(\ ,t)-u_{01})\|_{(-1/2)}(\partial\Omega)\\&\qquad+\|\partial_\nu(u_2(\ ,t)-u_{02})\|_{(-1/2)}(\partial\Omega)\le C(e^{Ct}\varepsilon^{t^{-2}/C}+e^{-\theta t}),\end{aligned}$$

where we also have made use of (9.4.10) and (9.4.12). To make the last two terms equal we let $t=(-\ln\varepsilon)^{1/3}/(2C)$. Denoting by $\varepsilon_1$ the norm of the difference of the Dirichlet-to-Neumann operators of equations (9.4.11), we conclude that

$$\varepsilon_1\le Ce^{-(-\ln\varepsilon)^{1/3}}/C.$$

By Theorem 5.2.3 we have

$$\|(c_2-c_1)+\lambda(a_{02}-a_{01})\|_\infty\le C(-\ln\varepsilon_1)^{-1/2}.$$

By using the bound on $\varepsilon_1$, we can replace $\varepsilon_1$ by $\varepsilon$, provided that $\frac{1}{2}$ is changed to $\frac{1}{6}$ and $C$ denotes possibly a larger constant. From the last estimate with two different $\lambda$ (say, $\lambda=1$ and $\lambda=2$) the bound (9.4.8) follows.

The proof is complete. □

At present, use of stabilization does not produce uniqueness results when measurements are implemented at a part of the boundary ($\Gamma\neq\partial\Omega$), because such results are not available for elliptic equations. This "local" case can be handled by again using the transform (9.2.1) and the recent method of Belishev for hyperbolic problems. We observe that the transform (9.2.1) applied to the parabolic equation (9.0.1) with time-independent coefficients and Theorem 9.2.1 imply that given the local (corresponding to $\gamma\subset\partial\Omega$) parabolic Dirichlet-to-Neumann map uniquely determine the local Dirichlet-to-Neumann map for the corresponding hyperbolic equation.

Indeed, the set of functions $g^0(x)\phi^*(t)$ with $g^0\in C_0^2(\gamma)$ and bounded $\phi^*\in C^2(\overline{\mathbb{R}}_+)$, $\phi^*(0)=\phi^{*'}(0)=0$, is complete in the subspace of functions in $C([0,T^*];H_{(1)}(\partial\Omega))$ that are zero at $\{t=0\}$ and outside $\gamma\times(0,T^*)$. A function $\phi$ determined by $\phi^*$ via (9.2.1) is analytic with respect to $t$, so a solution to the parabolic problem will be analytic with respect to $t$ as well. Then as above, the Dirichlet data for the hyperbolic problem generate the Dirichlet data for the parabolic problem whose solution is analytic with respect to time, so its Neumann data given on $\gamma\times(0,T)$ determine the Neumann data on $\gamma\times\mathbb{R}_+$ and due to uniqueness of the back transform $\partial_\nu u\to\partial_\nu u^*$ uniquely determine the hyperbolic Neumann data for any $T^*$. By applying Theorem 8.4.1, we obtain the following corollary.

**Corollary 9.4.4.** *Let $\gamma$ be any nonempty open part of $\partial\Omega$ and let the parabolic equation* (9.0.1) *have coefficients $a=1, b=0, c=0, f=0$.*

*Then the diffusion coefficient $a_0\in C^\infty(\overline{\Omega})$ is uniquely determined by the local parabolic Dirichlet-to Neumann map $\Lambda_l: g_0\to g_1$ on $\gamma\times(0,T)$*, supp $g\subset\gamma\times(0,T)$.

Similarly the known results about anisotropic hyperbolic equations [Be3], [KKL] lead to

**Corollary 9.4.5.** *Let $\gamma$ be any nonempty open part of $\partial\Omega$ and let the parabolic equation (9.0.1) have coefficients $a_0 = (\det a^*)^{1/2}$, $a = a_0 a^*$, where $a^*$ is a strictly positive symmetric $C^\infty(\overline{\Omega})$-matrix function, $b = 0$, $c = 0$, $\partial\Omega \in C^\infty$.*

*Then the local parabolic Dirichlet-to Neumann map uniquely determines $a^*$ modulo an isometry of the Riemannian manifold* $(\Omega, a)$.

In [KKL] there are also some partial results about recovery of more general hyperbolic equations modulo gauge transforms which similarly imply parallel results for parabolic inverse problems.

## 9.5 Discontinuous principal coefficient and recovery of a domain

The results and technique described above are applicable only to continuous coefficients of the principal part of an equation in divergent form, while in applications the discontinuous coefficient is quite important. As a typical example we will consider

$$a = a^0 + k\chi(Q^\bullet), \tag{9.5.1}$$

where $a^0, k$ are $C^2$-smooth functions and $\chi(Q^\bullet)$ is the characteristic function of the unknown domain $Q^\bullet$. In this section we describe uniqueness results for time-independent and time-dependent $Q^\bullet$ with given one or many lateral boundary measurements. To prove the first result we will make use of the transform (9.2.1) and of the Laplace transform, which enable one to apply uniqueness theorems for hyperbolic and elliptic equations. To prove the second result we utilize the method of products of singular solutions developed for elliptic equations in Section 5.7.

In the next theorem $\gamma$ is any open subset of the hyperplane $x_n = 0$ and $\Omega$ is the half-space $\{x_n < 0\}$ in $\mathbb{R}^n$.

**Theorem 9.5.1.** *Let $a_0 = 1$, and $Q^\bullet = D \times (0, T)$, where $D$ is a bounded Lipschitz $x_n$-convex domain in the half-space $\{x_n < 0\}$, $2 \le n$, intersecting the strip $\gamma \times \mathbb{R}_-$. Assume that $k$ is known constant and*

$$-1 < k < 0. \tag{9.5.2}$$

*Let the Dirichlet lateral data $g_0(x, t)$ be $g^0(x)\psi(t)$, where $\psi$ corresponds via* (9.2.1) *to a function $\psi^*(\tau)$ that is positive as well as its first-order derivative on some interval* $(0, \varepsilon)$, $|\psi_0(\tau)| \le C\exp(C\tau)$ *and* $g^0 = 1$ *on* $\Gamma$, $g^0 \in C_0^2(\partial\Omega)$. *Let $u$ be a solution to the parabolic problem* (9.0.1)–(9.0.3) *with $a$ given by* (9.5.1) *and with* $a_0 = 1, b = 0, c = 0, u_0 = 0$.

*Then $\partial_\nu u$ on $\gamma \times (0, T)$ uniquely determines $D$.*

PROOF. We consider the hyperbolic equation

$$\partial_\tau^2 u^* - \operatorname{div}((1 + k\chi(Q^\bullet))\nabla u^*) = 0 \quad \text{on } \Omega \times \mathbb{R}_+$$

with zero initial data and with the lateral data $u^*(x, \tau) = g^0(x)\psi^*(\tau)$ on $\partial\Omega \times \mathbb{R}_+$. $T^*$ is to be chosen later. Then $u(x, t)$ obtained from $u^*$ solves on $\Omega \times \mathbb{R}_+$ the parabolic problem (9.0.1)–(9.0.3) with $a_0 = 1, b = 0, c = 0$, and $a$ given by (9.5.1). As in Section 9.2, the Dirichlet lateral data for the parabolic problem are analytic with respect to $t$, due to our growth assumptions on $\psi^*$. Therefore, the solution of the parabolic problem is analytic with respect to $t$ as well, and the data $\partial_\nu u(x, t)$ of the inverse problem given on $\gamma \times (0, T)$ are uniquely determined on $\gamma \times \mathbb{R}_+$. Let us choose any $x'^0 \in \gamma$ such that the straight line through this point parallel to the $x_n$-axis intersects $\partial D$. Let $x^0 = (x'^0, x_n^0)$ be the intersection point with largest $x_n$. Since $\partial D$ is Lipschitz, there is a point of $\partial D$ near $x^0$ where there is a tangent (hyper-)plane to $\partial D$ whose normal is not perpendicular to $\gamma$. For brevity we can assume that this point is $x^0$. By the uniqueness Theorem 8.5.1 for inverse hyperbolic problems, $\partial D \cap V$ is uniquely identified for some neighborhood $V$ of $x^0$.

The Laplace transform

$$U(x, s) = \int_0^\infty u(x, t)e^{-st}dt$$

of the solution to the parabolic problem solves the elliptic problem with the parameter $s$,

$$-\operatorname{div}((1 + k\chi(D))\nabla U) - sU = 0 \quad \text{in } \Omega, s > 0.$$

The Dirichlet data of $U(, s)$ on $\partial\Omega$ are $g^0(x)\Psi(s)$ and not zero at some $s_1$. Moreover, $\partial_\nu U(, s)$ on $\partial\Omega\backslash D$ is uniquely determined because the solution of the parabolic problem is uniquely determined. We have proved already that two possible solutions $D_1, D_2$ of the inverse problem have a common piece of their boundaries. From our assumptions about $x_n$-convexity it follows that they are $i$-contact. Since both domains produce the same Cauchy data on $\Gamma$, they have to coincide by Theorem 4.3.2.

The proof is complete. □

To formulate the result for domains changing in time, we define the lateral boundary $\partial_x Q^\bullet$ of an open subset of the layer $\mathbb{R}^n \times (0, T)$ as the closure of $\partial Q^\bullet \cap \{0 < t < T\}$. We will say that $Q^\bullet$ is $x$-Lipschitz if $\partial_x Q^\bullet$ is Lipschitz.

**Theorem 9.5.2.** *Let $n \geq 2$. Suppose that $Q^\bullet$ is an open $x$-Lipschitz subset of $Q$ with $\partial_x Q^\bullet \cap \partial_x Q = \emptyset$ satisfying the conditions*

(9.5.3) *the sets $(Q\backslash\overline{Q^\bullet}) \cap \{t = \tau\}$ are connected when* $0 < \tau < T$

*and that scalar matrix $a^0$, a symmetric matrix $k$, and $b, c$ satisfy the conditions*

$$a^0, k \in C^2(\overline{\Omega}),\ a = a_0 + \chi(Q^\bullet)k, b = 0, c = 0, 0 < k,\ or\ -a_0 < k < 0,\ on\ \overline{\Omega} \tag{9.5.4}$$

*Then the lateral boundary data*

$$g_1 = \partial_\nu u \text{ on } \gamma \times (0, T)$$

*of a solution u to* (9.0.1)–(9.0.3) *(with* $\partial_x Q^\bullet, u_) = 0$*) given for any* $g_0 \in C_0^2(\partial\Omega \times (0, T))$ *that are zero outside* $\gamma \times (0, T)$ *uniquely determine* $Q^\bullet$. *If k is a scalar function, then it is uniquely determined on* $Q^\bullet$.

Detailed proofs for scalar $k$ are given in the paper of Elayyan and Isakov [ElI1]. They utilize the idea of using singular solutions of differential equations in inverse problems introduced and used by Isakov [Is3]. Since these proofs are quite technical, we will outline only the basic steps.

Let us assume that there are two domains $Q_1^\bullet$, $Q_2^\bullet$ with two $k_1, k_2$ producing the same boundary data. Let us denote by $Q_{3\theta}$ the connected component of the open set $\Omega \backslash (\overline{Q^\bullet_{1\theta}} \cup \overline{Q^\bullet_{2\theta}})$ whose boundary contains $\gamma$. Here $Q^\bullet_{j\theta}$ is defined as $Q^\bullet_j \cap \{t = \tau\}$. Let $Q_3$ be the union of $Q_{3\theta}$ over $0 < \theta < T$ and let $Q_4 = Q \backslash \overline{Q_3}$.

**Lemma 9.5.3.** *Under the conditions of Theorem* 9.5.2 *we have the following orthogonality relations:*

$$\int_{Q_1^\bullet} k_1 \nabla v_1 \cdot \nabla u_2^* = \int_{Q_2^\bullet} k_2 \nabla v_1 \cdot \nabla u_2^* \tag{9.5.5}$$

*for all solutions* $v_1$ *to the equation*

$$\partial_t v_1 - \operatorname{div}((a^0 + k_2\chi(Q_1))\nabla v_1) = 0 \text{ near } \overline{Q}_4, v_1 = 0 \text{ when } t < 0,$$

*and all solutions* $u_2^*$ *to the adjoint equation*

$$-\partial_t u_2^* - \operatorname{div}((a^0 + k_2\chi(Q_2))\nabla u_2^*) = 0 \text{ near } \overline{Q}_4, u_2^* = 0 \text{ when } T < t.$$

PROOF. As we did several times before, we subtract two equations (9.0.1) with the coefficients $a_2$ and $a_1$ and obtain for the difference $u$ of their solutions $u_2$ and $u_1$ the differential equation

$$\partial_t u - \operatorname{div}((a^0 + k_2\chi(Q_2^\bullet))\nabla u) = \operatorname{div}((k_2\chi(Q_2^\bullet) - k_1\chi(Q_1^\bullet))\nabla u_1) \text{ in } Q. \tag{9.5.6}$$

Since $u_2, u_1$ have the same lateral Cauchy data on $\gamma \times (0, T)$ by known uniqueness results (Theorem 3.3.10) they coincide on $Q_4$, so their difference $u$ is zero there.

Writing the definition (9.0.4) of the generalized solution to equation (9.5.6) with the test function $v_2^*$, we get

$$\int_Q (\partial_t u v_2^* + (1 + k_2\chi(Q_2^\bullet))\nabla u \cdot \nabla v_2^*) = -\int_Q (k_2\chi(Q_2^\bullet) - k_2\chi(Q_1^\bullet))\nabla u_1 \cdot \nabla v_2^*.$$

Applying again the definition (9.0.4) of the generalized solution to the equation for $v_2^*$ with the test function $u$, we complete the proof of the orthogonality relation (9.5.5) with $u_1$ instead of $v_1$. The final relation can be obtained by using the approximation of $v_1$ by $u_1$ exactly as was done for elliptic equations in Lemma 5.7.2. □

**Exercise 9.5.4.** Prove that the relation (9.5.5) with $v_1$ replaced by $u_1$ implies the general (9.5.5).

{*Hint*: make use of uniqueness of the continuation and of parabolic potentials to prove the Runge property of solutions of parabolic equations.}

Returning to the proof of Theorem 9.5.2, we assume that

$$Q_1^\bullet \neq Q_2^\bullet. \tag{9.5.7}$$

Then we may assume that there is a point $(x^0, t^0) \in \partial_x Q_1^\bullet \backslash \overline{Q_2^\bullet}$ that is contained as well in $\partial Q_3$. By considering $g_0 = 0$ when $t < t_0$ and using the translation $t \to t - t_0, x \to x - x_0$, we can also assume that $t_0 = 0, x_0 = 0$. Let us choose a ball $B \subset \mathbb{R}^n$ and a cylinder $Z = B \times (-\tau, \tau)$ such that $\overline{B} \subset \Omega$, $\overline{Z}$ is disjoint from $\overline{Q}_2$, and $(\partial_x Q_1) \cap \overline{Z}$ is a Lipschitz surface. By the Whitney extension theorem there is a $C^2(\overline{Q}_1 \cup \overline{Z})$-function $a_3$ coinciding with $a^0 + k_1$ on $Q_1$. We will extend $a_3$ onto $Q \backslash (\overline{Q}_1 \cup \overline{Z})$ as $a^0$.

To complete the proof we need the following modification of the orthogonality relations (9.5.5):

$$\int_{Q_1^\bullet} k_1 \nabla u_3 \cdot \nabla u_2^* = \int_{Q_2^\bullet} k_2 \nabla u_3 \cdot \nabla u_2^* \tag{9.5.8}$$

for any solution $u_3$ to the parabolic equation

$$\partial_t u_3 - \operatorname{div}(a_3 \nabla u_3) = 0 \text{ near } \overline{Q}_4, \quad u_3 = 0 \quad \text{when } t < 0$$

and for any solution $u_2^*$ to the parabolic equation from Lemma 9.5.3. The derivation of (9.5.8) from (9.5.5) is quite similar to the derivation of the relation (5.7.4) from the relation (5.7.2) in the elliptic case. It suffices to approximate $u_3$ by solutions $u_3$ of parabolic equations with $a_3$ replaced by $a_{3,m}$ and with the same initial and lateral boundary Dirichlet data as $u_3$. The sequence of the coefficients $a_{3,m}$ is chosen so that they are uniformly bounded from zero and from infinity, equal to $a_3$ in a neighborhood of $\overline{Q}_4$ depending on $m$, and pointwise convergent to $a_3$. The orthogonality relation (9.5.5) is valid with $v_1 = u_{3,m}$, and the relation (9.5.8) is obtained by passing to the limit when $m \to \infty$.

To obtain a contradiction with the initial assumption, we will use as $u_3$ and $u_2^*$ the fundamental solutions $K^+$ and $K^-$ of the forward and backward Cauchy problems for the parabolic equations with coefficients $a_3$ and $a_2$ with poles at the points $(y, 0), (y, \tau)$, where $y$ is outside $\overline{Q}_4$ and is close to the origin. To obtain bounds of integrals it is convenient to use new variables. We can assume that the direction $e_n$ of the $x_n$-axis coincides with the interior unit normal to $\partial_x Q_1^\bullet \cap \{t = 0\}$. This surface near the origin is the graph of a Lipschitz function $x_n = q_1(x_1, \dots, x_{n-1}, t)$. The substitution

$$x_m = x_m^0, m = 1, \dots, n-1, x_n = x_n^0 + q_1(0, t), t = t^0$$

transforms the equations into similar ones (with additional first-order differential operators with respect to $x_n^0$). The domains $Q_j^\bullet$ are transformed into similar domains, with the additional property that the points $(0, t)$, $0 < t < T$, are in $\partial_x Q_1^\bullet$.

Since $q_1$ is a Lipschiz function we may assume that $(0, \dots, 0, 1)$ is the interior normal to its graph at the origin. Further on, we will drop the index 0.

The singular (fundamental) solutions to be used have the following structure:

$$K^+ = K_1^+ + K_0^+, K^- = K_1^- + K_0^-, \tag{9.5.9}$$

where the first terms are the so-called parametrices

$$K_1^+(x,t;y,\tau) = C(deta_3^{-1}(y)(t-\tau))^{-n/2}\exp(-a^{-1}(y)(x-y)$$
$$\cdot(x-y)(/(4(t-\tau))),$$
$$K_1^-(x,t;y,\tau) = C(a^0(y))^{1/2}((\tau-t))^{-n/2}\exp(-|x-y|^2/(4a^0(y)(\tau-t))),$$

and $K_0^+$, $K_0^-$ are remainders with weaker singularities satisfying the bounds

$$|\nabla_x^j K_0^+(x,y;t,\tau)| \le C|t-\tau|^{(\lambda-n-j)/2} exp(-C|x-y|^2/(t-\tau)),$$

$$|\nabla_x^j K_0^-(x,y;t,\tau)| \le C|t-\tau|^{(\lambda-n-j)/2} exp(-C|x-y|^2/(\tau-t)), j = 0, 1.$$

Letting in (9.5.8) $u_3 = K^+(\,;y,0)$, $u_2^* = K^-(\,;y,\tau)$, splitting $K^+$, $K^-$ according to (9.5.9), and breaking the integration domain $Q_1^\bullet$ into $Q_1^\bullet \cap Z$ and its complement, we obtain

$$I_1 = I_2 + I_3, \tag{9.5.10}$$

where

$$I_1 = \int_{Q_1^\bullet \cap Z} k_1 \nabla_x K_1^+ \cdot \nabla_x K_1^-,$$
$$I_2 = -\int_{Q_1^\bullet \setminus Z} k_1 \nabla_x K^+ \cdot \nabla_x K^- + \int_{Q_2^\bullet} k_2 \nabla_x K^+ \cdot \nabla_x K^-,$$

and

$$I_3 = \int_{Q_1^\bullet \cap Z} k_1(\nabla_x K_1^+ \cdot \nabla_x K_0^- + \nabla_x K_0^+ \cdot \nabla_x K_1^- + \nabla_x K_0^+ \cdot \nabla K_0^-).$$

To obtain the relation (9.5.10) we first take $y = (0, \dots, 0, -\delta)$ and then let $\delta \to 0$. The passage to the limit can be justified (when $\tau > 0$ is fixed) by using the Lebesgue dominated convergence theorem and bounds on $x$-derivatives of the fundamental solutions, which follows from the given formulae for parametrices and from known results (see the book by Friedman [Fr]). After the passage to the limit, the integrals in (9.5.10) became functions of $\tau$ only.

**Lemma 9.5.5.** *We have*

$$\tau^{-n/2}/C \le |I_1|,$$
$$|I_2| \le C,$$
$$|I_3| \le C\tau^{-n/2+\lambda/2},$$

*where constant $C$ does not depend on $\tau$.*

PROOF. From (9.5.9) we have

$$\nabla_x K_1^+(x,t;0,0) = -K_1^+(x,t;0,0)/(2t)a_1^{-1}(0)x,$$
$$\nabla_x K_1^-(x,t;0,0) = -K_1^-(x,t;0,0)/(2ta_0(0))x.$$

Since $a_1 = a^0 + k_1$ on $Q_1$ and $a^0$ is scalar, the matrices $k_1(0), a_1(0), a_1^{-1/2}(0)$ commute, using in addition their symmetry we will have

$$\begin{aligned}&|k_1(x)(a_1^{-1}(0)x)) \cdot (a_0^{-1}(0)x)|\\ &\quad = |k_1(0)a_1^{-1/2}(0)a_0^{-1/2}(0))x \cdot (a_1^{-1/2}(0)a_0^{-1/2}(0)x)\\ &\qquad + (k_1(x) - k_1(0))(a_1^{-1}(0)x \cdot (a_0^{-1}(0)x)|\\ &\quad \geq |x|^2/C - C|x|^3 \geq |x|^2/C,\ x \in B\end{aligned}$$

where we used the trangle inequality and condition (9.5.4) on $k = k_1$ and choose $B$ to have small radius. By our regularity assumptions there is an open cone $\mathcal{C}$ in $\mathbb{R}^n$ with vertex at the origin and axis $(0,\ldots,0,s)$, $0 < s < \varepsilon_0$ that is contained in $Q_1^\bullet$. Hence,

$$\begin{aligned}|I_1| &\geq 1/C \int_{\mathcal{C}\times(0,\tau)} |x|^2(t(\tau-t))^{-n/2-1} exp(-|x|^2/(mt) - |x|^2/(m(t-\tau)))dtdx\\ &\geq 1/C \int_{\mathcal{C}}\int_0^\tau |x|^2(t(t-\tau))^{-n/2-1} exp(-|x|^2\tau/(mt(\tau-t)))dtdx\end{aligned}$$

where $m$ is the smallest of the eigenvalues of $a_1, a^0$. Using the inequality

$$1/(t\tau) \leq 1/(t(\tau-t)) \leq 2/(t\tau) \text{ when } 0 < t < \tau/2$$

and the previous bound we obtain

$$\begin{aligned}|I_1| &\geq 1/C \int_{\mathcal{C}}\int_0^{\tau/2} |x|^2(t\tau)^{-n/2-1} exp(-2|x|^2/(mt))dtdx\\ &= 1/C\tau^{-n/2-1}\int_0^\varepsilon \rho^2(\int_0^{\tau/2} t^{-n/2-1} exp(-2\rho^2/(mt))dt)\rho^{n-1}d\rho\end{aligned}$$

where we have used the polar coordinates $\rho = |x|$ in $\mathbb{R}^n$. Using the substitutiton $w = 2\rho^2/(mt)$ in the last integral we yield

$$\begin{aligned}|I_1| &\geq 1/C\tau^{-n/2-1}\int_0^\varepsilon \rho(\int_{4\rho^2/(m\tau)} w^{n/2-1}e^{-w})d\rho\\ &\geq 1/C\tau^{-n/2-1}\int_0^{4\varepsilon^2/(m\tau)}(\int_0^{\sqrt{m\tau w}/2} \rho d\rho)w^{n/2-1}e^{-w}dw\end{aligned}$$

where we replaced the integration domain by the smaller one $\{0 < \rho < \varepsilon, 4\rho^2/(m\tau) < w < 4\varepsilon^2/(m\tau)\}$ and interchanged order of integration. Calculating the last inner integral we get

$$|I_1| \geq 1/C\tau^{-n/2}\int_0^{4\varepsilon^2/(m\tau)} w^{n/2}e^{-w}dw.$$

The proof of the first bound of Lemma 9.5.5 is complete.

The second bound is obvious, since the fundamental solution is bounded away from singularities.

The third bound can be obtained similarly to the first one by using the bounds (9.5.9) on $K_0^+$, $K_0^-$. Of course, now we obtaining bounds from above. We will not give a complete argument, only some outlines. One of the integrals to be bounded looks like that one for $I_1$, it is less than the integral of

$$|x|(\tau - t)^{-n/2-1}t^{\lambda/2-n/2-1/2}e^{-|x|^2/(mt)}e^{-|x|^2/(m(\tau-t))}$$
$$\le C|x|^{2+\lambda}(t(\tau - t))^{-n/2-1}e^{-|x|^2/(m_1 t)}e^{-|x|^2/(m_1(\tau-t))}$$

when $0 < m < m_1$. Here we used that $|x|^{1+\lambda}t^{-(1+\lambda)/2}exp(-(m_1 - m)|x|^2/(mm_1 t)) < C$. After these remarks upper bounding $I_3$ repeats lower bounding $|I_1|$. □

To complete the proof of Theorem 9.5.2 we first show that domains coincide. From (9.5.10) and Lemma 9.5.5 we have

$$\tau^{-n/2} \le C(1 + \tau^{-n/2+\lambda/2})$$

and letting $\tau \to 0$ we will have a contradiction. The contradiction shows that the initial assumption is wrong, so $Q_1^\bullet = Q_2^\bullet$.

When the domains coincide, one can prove equality of scalar $k_j$ on $Q_1^\bullet$ by using the methods of sections 5.1,9.4.

The first claim is that

$$k_1 = k_2,\ \nabla k_1 = \nabla k_2 \text{ on } \partial_x Q^\bullet. \tag{9.5.11}$$

As in the proof for domains we assume the opposite. Then we can assume that the origin $0 \in \partial_x Q_1^\bullet$ and $\varepsilon < k_2 - k_1$ on some ball $B$ centered at the origin. As above there is function $a_4 \in C^2(\mathbb{R}^n)$ conciding with $a_2$ on $Q_1^\bullet$. By repeating the proof of (9.5.8) we conclude that

$$\int_{Q_1^\bullet} (k_2 - k_1)\nabla u_3 \cdot \nabla u_4^* = 0$$

for all solutions $u_3$ to the equation $\partial_t u_3 - \operatorname{div}(a_3 \nabla u_3) = 0$ near $\overline{Q_1^\bullet}$ which are zero when $t < 0$ and for all solutions $u_4^*$ to the equation $\partial_t u_4^* + \operatorname{div}(a_4 \nabla u_4^*) = 0$ near $\overline{Q_1^\bullet}$ which are zero when $T < t$. Repeating the proof of uniqueness for domains we will obtain a contradiction. Hence we have the first equality (9.5.11). Repeating this argument with use of normal derivatives of fundamental solutions we prove the second equality (9.5.11).

To conclude it suffices to show that $k_1 = k_2$ on the intersection $\Omega_0$ of all $Q_1^\bullet \cup \{t = \theta\}$ over all $\theta \in (0, T)$. Let $Q_0 = \Omega_0 \times (0, T)$. By using local representations of $\partial_x Q_1^\bullet$ and using that $min$ of a family of $Lipschitz$ functions is Lipschitz one can show that (at least for small T) $\Omega_0$ is Lipschitz. Due to (9.5.11) and to time independence, $k_1 = k_2$ on $Q \backslash \Omega_0 \times (0, T)$. We can assume that $T$ is small, since increasing $T$ shrinks $Q_0$. Letting $a_5 = a_0 + \chi(\Omega_0)k_1, a_6 = a_0 + \chi(\Omega_0)k_2$

and adjusting the derivation of (5.7.3) from (5.7.2) to the parabolic case we obtain from Lemma 9.7.4 that

$$\int_{Q_0} (k_2 - k_1)\nabla_x u_5 \cdot \nabla_x u_6^* = 0, \tag{9.5.12}$$

for all solutions $u_5$ to the equation $\partial_t u_5 - \operatorname{div}(a_5 \nabla_x u_5) = 0$ near $\overline{Q_0}$ which are zero when $t < 0$ and for all solutions $u_6^*$ to the equation $\partial_t u_6^* + \operatorname{div}(a_6 \nabla_x u_6^*) = 0$ near $\overline{Q_0}$ which are zero when $T < t$.

We claim that (9.5.12) implies equality of the lateral Dirichlet-to Neumann maps for parabolic equations with coefficients $a_5, a_6$.

Indeed, let $u_5$ and $u_6$ be solutions of these equations in $Q$ with zero initial data and with the same (smooth) lateral Dirichlet data $g_0$ supported in $\gamma \times (0, T)$. By subtracting these equations and letting $u = u_6 - u_5$ we yield

$$\partial_t u - \operatorname{div}(a_6 \nabla_x u) = \operatorname{div}((a_5 - a_6)\nabla_x u_5) \text{ on } Q.$$

From the definition of a generalized solution ( before Theorem 9.1) we have

$$-\int_{\partial\Omega\times(0,T)} a^0 \partial_\nu u v + \int_Q (a_6 \nabla_x u \cdot \nabla_x v - u \partial_t v) = \int_Q (a_6 - a_5)\nabla_x u_5 \cdot \nabla_x v \tag{9.5.13}$$

for any function $v \in L_2(0, T; H_{(1)}(\Omega))$ with $\partial_t v \in L_2(Q)$, $v = 0$ on $\Omega \times \{T\}$. Since coefficients $a_5, a_6$ it follows ( by forming finite differences in time) from Theorem 9.1 that $\partial_t u_5, \partial_t u_6 \in L_2(Q)$, and $\partial_t u_6^* \in L_2(Q)$ for any solution to the adjoint equation $\partial_t u_6^* + \operatorname{div}(a_6 \nabla_x u^*) = 0$ in $Q$ with zero data at $t = T$ and smooth lateral Dirichlet data on $\partial_x Q$. Using again the definition of a(weak) solution to the adjoint equation with the test function $u$ we obtain

$$\int_Q (a_6 \nabla u_6^* \cdot u - u_6^* \partial_t u) = 0,$$

because $u = 0$ on $\partial_x Q$. Letting $v = u_6^*$ in (9.5.13) and using the last integral equality as well as (9.5.12) we yield

$$\int_{\partial\Omega\times(0,T)} a^0 \partial_\nu u u_6^* = 0.$$

Since $u_6^*$ on $\partial\Omega \times (0, T)$ is an arbitrary smooth function, $\partial_\nu u = 0$ on $\partial\Omega \times (0, T)$. So $a_5, a_6$ generate the same lateral Dirichlet-to Neumann map. Now by using stabilization of solutions of parabolic problems for large $t$ as in the proof of Theorem 9.4.1 we conclude that the elliptic equations

$$-\operatorname{div}(a_j \nabla v_j) + s v_j = 0, \text{ in } \Omega, \ \ j = 5, 6,$$

have the same Dirichlet-to Neumann maps. By Theorem 5.7.1, $a_5 = a_6$ and $k_1 = k_2$.

Finally we prove similar results for impenetrable domains for single boundary measurements which in particular guarantee that under natural assumptions voids are uniquely identified by their exterior thermal field.

**Theorem 9.5.6.** *Let $u$ solve the parabolic equation (9.0.1) with $f = 0$ in $Q\backslash Q^\bullet$ where $Q^\bullet$ is a subdomain of $Q$ satisfying the conditions of Theorem 9.5.2. Let $u$ have the initial condition $u_0 = 0$ on $\Omega \times \{0\}\backslash \overline{Q}^\bullet$ and the Dirichlet condition (9.0.3) with nonnegative $g_0 \in C^2(\partial\Omega \times [0, T])$ which is not identically zero on $\partial\Omega \times (0, \theta)$ for any $\theta > 0$. Let*

(9.5.14)

$$u = 0 \text{ on } \partial_x Q^\bullet \ (D) \ \text{ or } \partial_{\nu(a)} u = 0 \text{ on } \partial_x Q^\bullet \text{ and } Q^\bullet = \Omega^\bullet \times (0, T) \ (N).$$

*Then the additional Neumann data (9.0.6) for any open subset $\gamma$ of $\partial\Omega$ uniquely determine $Q^\bullet$.*

We will prove this result by the Schiffer's method which is used in section 6.3 to show uniqueness of a soft obstacle.

PROOF. Let us assume that there are two different domains $Q_1^\bullet$, $Q_2^\bullet$ producing the same lateral boundary data. We will introduce the domains $Q_3$, $Q_4$ as in the proof of Theorem 9.5.2. We may assume that $Q_2^\bullet$ is not contained in $Q_1^\bullet$, hence as in the proof of Theorem 6.3.1 there is a connected component $Q_0$ of $Q_4\backslash\overline{Q}_1^\bullet$. The lateral boundary $\partial_x Q_0 = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \subset \partial_x Q_1^\bullet$ and $\Gamma_2 \subset \partial_x Q_1^\bullet \cap \partial_x Q_3$. Since $u_1, u_2$ satisfy the same parabolic equation in $Q_3$ by uniqueness in the lateral Cauchy problem (Theorem 3.3.10) $u_1 = u_2$ on $Q_3$. Hence $u_1 = u_2 = 0$ on $\Gamma_2$ in case $(D)$ and $\partial_{\nu(a)} u_1 = 0$ on $\Gamma_2$ in case $(N)$.

Replacing $u_1$ by $e^{\tau t} v_1$ we will have for $v_1$ the parabolic equation (9.0.1) with $c$ replaced by $c + \tau a_0$. To show that $v_1 = 0$ on $Q_0$ we will make use of the definition of weak solution in $Q_0$:

$$\int_{Q_0} (a_0 \partial_t v_1 \phi + (a\nabla v_1) \cdot \nabla\phi + (b \cdot \nabla v_1 + (c_0 + \tau a_0) v_1)\phi)$$
$$- \int_{\Gamma_1} (a\nabla v_1) \cdot \nu\phi - \int_{\Gamma_2} (a\nabla v_1) \cdot \nu\phi = 0$$

for any test function $\phi \in L_2(Q\backslash Q_1^\bullet)$ with $\nabla\phi \in L_2(Q\backslash Q_1^\bullet)$, $\phi \in H_{(1/2)}(Q\backslash Q_1^\bullet)$. Letting $\varphi = v_1$, observing that $(\partial_t v_1) v_1 = 1/2 \partial_t (v_1^2)$, and integrating by parts in the first integral with use of zero boundary conditions on $\Gamma_1, \Gamma_2$ and at $t = 0$ we obtain

$$\int_{\partial Q_0 \cap \{t=0\}} 1/2 a_0 v_1^2 + \int_{Q_0} (a\nabla v_1 \cdot \nabla v_1 + b \cdot \nabla v_1 v_1 + (a_0 \tau + c - \partial_t a_0/2) v_1^2) = 0.$$

Using the elementary inequality $-\varepsilon|\nabla v_1|^2 - C(\varepsilon) v_1^2 \leq b \cdot \nabla v_1 v_1$ and positivity of the matrix $a$ we yield

$$\int_{Q_0} (\varepsilon_0 |\nabla v_1|^2 - \varepsilon|\nabla v_1|^2 + (a_0\tau + c - \partial_t a_0/2 - C(\varepsilon)) v_1^2 \leq 0.$$

Choosing $\varepsilon < \varepsilon_0$ and then $\tau$ to be large, we obtain that $v_1 = 0$ in $Q_0$. By uniqueness of the continutation from $Q_0$ (Theorem 3.3.10) we have $v_1 = 0$, $u_1 = 0$ on $Q_3 \cap \{T_1 < t < T_2\}$ where $T_1 = inf t$ and $T_2 = sup t$ over $(x, t) \in Q_0$. Hence $g_0 = 0$

on $\partial\Omega \times (T_1, T_2)$. In case $(N)$ $T_1 = 0$ and we have a contradiction with conditions of Theorem 9.5.6 on $g_0$. In case $(D)$ $u_1$ is nonnegative by Theorem 9.2 and hence by the second part of Theorem 9.2 we have $u_1 = 0$ on $(Q\backslash Q_1^\bullet) \cap \{0 < t < T_2\}$, $g_0 = 0$ on $\partial\Omega \times (0, T_2)$ and we have a contradiction again.

The proof is complete. □

Instead of smooth function $g_0$ one can use the Dirac delta-function concentrated at a point of $\partial\Omega \times \{0\}$. The proof is valid in this case with minor technical adjustments regarding definition of a weak soltion with less regular Dirichlet data. Vessella [Ve] obtained a logarithmic stability estimate for this problem.

## 9.6 Nonlinear equations

In this section we consider recovery of the terms $a_0(x, u)$, $c(x, t, u)$ of the nonlinear parabolic equation

(9.6.1) $$a_0(x, u)\partial_t u - \Delta u + c(x, t, u) = 0 \text{ in } Q$$

from two natural analogues of the Dirichlet-to-Neumann map for this equation. About the nonlinear terms we assume that

(9.6.2) $$0 < a_0 \text{ on } \Omega \times \mathbb{R}, 0 \leq \partial_u c, c(x, t, 0) = 0, \\ a_0, \partial_u a_0, \partial_u^2 a_0, c, \partial_u c, \partial_u^2 c \text{ are in } L_\infty(Q \times [-U, U])$$

for any finite $U$.

By using maximum principles and upper and lower solutions and Theorem 9.1 as in Section 5.6 for elliptic equations, one can show that for any $C^2$-smooth compatible data $u_0, g_0$ there is a unique solution $u(x, t; u_0, g_0)$ in $H_{2,1;2}(Q) \cap C(\overline{Q})$. Therefore, we have a well-posed direct (boundary value) problem for the nonlinear parabolic equation (9.6.1), and the following functions are well-defined:

$$u_T = u \text{ on } \Omega \times \{T\}, \quad g_1 = \partial_\nu u \quad \text{on } \partial\Omega \times (0, T).$$

Let $E_{par} = \{(x, t, u) : (x, t) \in Q,\ u = u(x, t; \theta, \theta) \text{ for some constant } \theta\}$. As for elliptic equations values of solutions $u(x, t)$ to the nonlinear initial boundary value problem for the parabolic equation (9.6.1) with all possible data $u_0, g_0$ under conditions (9.6.2) do not necessarily cover $\mathbb{R}$, and we hope to have uniqueness at most on $E_{par}$.

**Theorem 9.6.1.** *Assume that*

(9.6.3) $$a_0 = 1.$$

*Then the complete Dirichlet-to-Neumann map $\Lambda : (g_0, u_0) \to (g_1, u_T)$ uniquely determines $c$ on $E_{par}$.*

PROOF. We will adapt to the inverse parabolic problem the proof of Theorem 5.6.1.

Let $c^*(x, t; \theta)$ be $\partial_u c(x, t; u(x, t; \theta, \theta))$. We consider the linear parabolic equation

$$\partial_t v - \Delta v + c^* v = 0 \text{ on } Q \tag{9.6.4}$$

and let $\Lambda^*$ be the complete Dirichlet-to-Neumann map for this linear equation. □

**Lemma 9.6.2.** $\Lambda$ *uniquely determines* $\Lambda^*$. *Moreover, the function* $v = \partial_\theta u(;\theta, \theta)$ *is well-defined, satisfies the differential equation* (9.6.4), *and has initial and lateral boundary data* 1.

PROOF. As in the proof of Lemma 5.6.2 we take any pair $(g_0^*, u_0^*)$ of the admissible lateral and initial data. Let us subtract the equations for $u(;\theta + \tau g_0^*, \theta + \tau u_0^*)$ and for $u(;\theta, \theta)$, divide the result by $\tau$, and denote by $v(;\tau)$ the finite difference $(u(;\theta + \tau g_0^*, \theta + \tau u_0^*) - u(;\theta, \theta))/\tau$. Using Taylor's formula as in Section 5.6 and the regularity assumptions (9.5.2) on $c$, we obtain the linear differential equation

$$\partial_t v(\,;\tau) - \Delta v(\,;\tau) + c^*(\,;\theta, \tau) v(\,;\tau) = 0 \text{ on } Q, \tag{9.6.5}$$

where

$$c^*(\,;\theta, \tau) = \int_0^1 \partial_u c(\,;(1-s)u(\,;\theta, \theta) + su(\,;\theta + \tau g^*, \theta + \tau u_0^*))ds,$$

and the initial and lateral Dirichlet boundary data

$$v(\,;\tau) = u_0^* \text{ on } \Omega \times \{0\} \text{ and } v(\,;\tau) = g_0^* \text{ on } \partial\Omega \times (0, T). \tag{9.6.6}$$

According to conditions (9.6.2), we have $\|c^*\|_\infty(Q) < C$, where $C$ does not depend on $\tau$, so Theorem 9.1 gives that $\|v(;\tau)\|_\infty(Q) < C$. More detail is given for elliptic equations in section 9.5. Now, the formula for $c^*$ and conditions (9.6.2) imply that $c^*(;\theta, \tau) = c^*(;\theta) + r_1$ where $\|r_1\|_\infty \le C\tau$. Finally, the bounds on $v(;\tau)$ and on $r_1$ combined with Theorem 9.1 permit us to conclude that solutions to the parabolic boundary value problem (9.6.5)–(9.6.6) have a $H_{2,1;p}(Q)$-limit $v$ as $\tau \to 0$. The function $v$ apparently satisfies equation (9.6.4) and the initial and lateral boundary conditions (9.6.6). Since we are given the complete Dirichlet-to-Neumann map for equation (9.6.1), we are given $\partial_\nu v(;\tau)$ on $\partial\Omega \times (0, T)$ and $v(;\tau)$ on $\Omega \times \{T\}$. The convergence in $H_{2,1;2}(Q)$ and the trace theorems imply that these functions are convergent in $L_2((0, T); H_{(1/2)}(\partial\Omega))$ and in $L_2(\Omega)$ respectively. So the complete Dirichlet-to-Neumann map $\Lambda^*$ for the linearized equation (9.6.5) is uniquely defined by $\Lambda$.

The proof is complete. □

**Lemma 9.6.3.** *The complete Dirichlet-to-Neumann map for the linear equation* (9.6.4) *uniquely identifies its coefficient* $c^*$.

PROOF. Assume that we have two different coefficients $c_1^*$ and $c_2^*$ generating the same complete Dirichlet-to-Neumann map $\Lambda^*$. As in Section 5.3, subtracting equations (9.6.4) with the coefficients $c_2^*$ and $c_1^*$, we obtain

$$\partial_t u - \Delta u + c_2^* u = (c_1^* - c_2^*)u_1 \text{ on } Q,\ u = 0 \text{ on } \partial Q,\ \partial_\nu u = 0 \text{ on } \partial_x Q,$$

where $u_1, u_2$ are solutions to the equations (9.6.4) with the coefficients $c_1^*, c_2^*$, which have the same initial and lateral Dirichlet boundary data, and $u = u_2 - u_1$. The function $u$ and its normal derivatives are zero at the boundary, because both coefficients produce the same Dirichlet-to-Neumann map. Let $v_2$ be any solution to the adjoint equation

$$\partial_t v - \Delta v + c_2^* v = 0 \text{ in } Q, \quad v \in H_{2,1;2}(Q).$$

Using the identity (9.0.4) for the equation for $u$ and then the same identity for the equation for $v_2$, we conclude that

$$\int_Q (c_1^* - c_2^*) u_1 v_2 = 0$$

for any solution $u_1$ of the parabolic equation with the coefficient $c_1$ and any solution to the above-mentioned adjoint equation. To prove uniqueness it suffices to show that products of solutions of these equations are complete in $L_1(Q)$. We will derive this result from Theorem 5.3.3.

To check condition (5.3.8) of this theorem, we choose any $\xi(0) \in \mathbb{R}^{n+1}$. Due to the invariancy of the principal symbols $P_1(\zeta) = i\zeta_{n+1} + \zeta_1^2 + \cdots + \zeta_n^2$, $P_2(\zeta) = -i\zeta_{n+1} + \zeta_1^2 + \cdots + \zeta_n^2$ with respect to rotations in $\mathbb{R}^n$, we will assume that $\xi(0) = (\xi_1(0), 0, \ldots, 0, \xi_{n+1}(0))$. We let $\Xi_0$ be $\{\xi_1^2 + \cdots + \xi_n^2 \neq 0\}$. The equations $P_1(\zeta(1)) = P_2(\zeta(2)) = 0$ for the vectors

$$\zeta(1) = (\zeta_1, \zeta_2, 0, \ldots, 0, \zeta_{n+1}),$$
$$\zeta(2) = (\xi_1(0) - \zeta_1, -\zeta_2, 0, \ldots, 0, \xi_{n+1}(0) - \zeta_{n+1})$$

can be transformed into

$$\zeta_1^2 + \zeta_2^2 + i\zeta_{n+1} = 0, \xi_1^2(0) - 2\xi_1(0)\zeta_1 - i\xi_{n+1}(0) = 0.$$

These algebraic equations have the solutions

$$\zeta_1 = \xi_1(0)/2 - i\xi_{n+1}(0)/(2\xi_1(0)), \zeta_2 = iR,$$
$$\zeta_{n+1} = \xi_{n+1}(0)/2 + i(\xi_1^2(0)/4 - \xi_{n+1}^2(0)/(2\xi_1(0))^2 - R^2).$$

We have

$$P_1(\xi + \zeta) = 2\zeta_1\xi_1 + \cdots + 2\zeta_n\xi_n + i\xi_{n+1} + \xi_1^2 + \cdots + \xi_n^2,$$
$$\text{so } \tilde{P}_1^2(\xi + \zeta(1)) \geq |\partial_{\xi_2} P_1(\xi + \zeta(1))|^2 = 4|\xi_2 + \zeta_2(1)|^2 \geq 4R^2.$$

Similarly, $\tilde{P}_2^2(\xi + \zeta(2)) \geq 4R^2$. Therefore, the conditions of Theorem 5.3.3 are satisfied, and the products $u_1 v_2$ are complete in $L_1(Q)$.

The proof is complete. □

END OF THE PROOF OF THEOREM 9.6.1. Now we will complete the proof of Theorem 9.6.1 quite quickly.

By Lemmas 9.6.2, 9.6.3 the function $\partial_u c(x, t, u(x, t; \theta, \theta))$ is uniquely identifiable. Then by Lemma 9.6.2 the function $v = \partial_\theta u(;\theta, \theta)$ can be uniquely found as a solution to the linear parabolic equation (9.6.4) with known coefficient and initial and lateral boundary data. Since $u(;0,0) = 0$ we can uniquely determine $u(;\theta,\theta)$. By the positivity principle $v$ is strictly positive on $\overline{Q}$. So the equation $u = u(x,t;\theta,\theta)$ has the unique solution $\theta(u;x,t)$ provided $(x,t,u) \in E_{par}$. Plugging this $\theta$ into $c^*$, we find $\partial_u c(x,t;u)$. Since $c(x,t;0) = 0$ we uniquely determine $c(x,t;u)$. □

We introduce the set $E_p = \{(x,u) : u_*(x) \le u \le u^*(x)$ when $0 < t < T\}$, where $u_*(x)$ is $\inf u(x,t)$ over all admissible (regular bounded) $g$ and over $t \in (0,\infty)$ (and $u^*$ is sup). The following result in a weaker form was obtained by Isakov in the paper [Is10].

**Theorem 9.6.4.** *Let* $2 \le n$ *and let* $Q = \Omega \times (0,+\infty)$. *Then the lateral Dirichlet-to-Neumann map* $\Lambda_l$ *uniquely determines* $c = c(x,u)$ *on* $E_p$. *If in addition* $3 \le n$, *then* $\Lambda_l$ *also uniquely determines* $a_0$ *on* $E_p$.

Before proving this result, we recall some known results about the asymptotic behavior of a solution $u(x,t)$ of the first initial boundary value problem for the nonlinear parabolic equation (9.6.1) when $t$ is large. We refer for a proof to the paper of Guidetti [Gu]. Assume that the lateral Dirichlet data are

$$g_0(x,t) = g^0(x) + g^1(x)t^{-1} + g^2(x)t^{-2} \text{ when } 1 \le t, \tag{9.6.7}$$

where $g^0, g^1, g^2$ are $C^2(\Omega)$-functions, with $g^1, g^2 \le 0$. Let $u^0, u^1, u^2$ be solutions to the following elliptic (stationary) Dirichlet problems:

$$-\Delta u^0 + c(x,u^0) = 0 \text{ in } \Omega,\ u^0 = g^0 \text{ on } \partial\Omega, \tag{9.6.8}$$

$$-\Delta u^1 + \partial_u c(x,u^0) = 0 \text{ in } \Omega,\ u^1 = g^1 \text{ on } \partial\Omega, \tag{9.6.9}$$

$$-\Delta u^2 + \partial_u c(x,u^0)u^2 = -\frac{1}{2}\partial_u^2 c(x,u^0)(u^1)^2 + a_o(x,u^0)u^1(x) \text{ in } \Omega,$$

$$u^2 = g^2 \text{ on } \partial\Omega. \tag{9.6.10}$$

From known results about parabolic initial boundary value problems we can conclude that a solution $u(x,t)$ to (9.0.1$_n$), (9.0.2), (9.0.3) with $g$ as given in (9.6.7) admits the following representation:

$$u(x,t) = u^0(x) + u^1(x)t^{-1} + u^2(x)t^{-2} + w(x,t), \tag{9.6.11}$$

where $t^2\|w(,t)\|_{1,p}(\Omega) \to 0$ for any $p > 1$ as $t \to \infty$.

Proof of Theorem 9.6.4. First we will make use of $g_0 = g^0$. From (9.6.8), (9.6.11) it follows that $u(t) \to u^0$ in $H_{1,p}(\Omega)$ as $t$ goes to infinity. By the trace theorems we have then $\partial_\nu u(,t) \to \partial_\nu u^0$ in the space $H_{(-1/2)}(\partial\Omega)$. According to the condition of Theorem 9.6.4, $\partial_\nu u(,t)$ on $\partial\Omega$ is given for any $t$. Therefore, we are given $\partial_\nu u^0$ on $\partial\Omega$ for any (smooth) Dirichlet data $g^0$. So we are given the Dirichlet-to-Neumann map for the semilinear elliptic equation (9.6.8), which by Theorem 5.6.1 uniquely determines $c(x,u)$ on $E$ (the set defined similarly to $E_p$ for the elliptic equation (9.6.8); see Section 5.6). We will explain next that $E_p = E$.

Indeed, in Section 5.6 we observed that due to monotonicity properties, $E$ can be defined by using only constant Dirichlet data on $\partial\Omega$. By using a stabilization argument, it is easy to see that $E \subset E_p$. We will show the inverse inclusion. Let $g_0$ be any admissible Dirichlet boundary data for the parabolic problem and $g_0 < C$ (a constant). Denote by $u$ the solution of the nonlinear parabolic equation with zero Cauchy data on $\Omega \times \{0\}$ and lateral Dirichlet data $g_0$ on $\partial\Omega \times (0, +\infty)$. Consider the new Dirichlet lateral boundary data $g^+(x,t) = \psi(t)G$, where $\psi$ is a smooth nondecreasing function that is 0 when $t < -1$ and 1 when $1 < t$. Let $u^+$ be the solution of the same equation on $\Omega \times (-1, +\infty)$ with zero Cauchy data on $\Omega \times \{-1\}$ and lateral Dirichlet data $g^+$ on $\partial\Omega \times (-1, +\infty)$. Since the initial and lateral boundary data for $u$ are less than for $u^+$ by the positivity principle we have $u \le u^+$ on $Q$. On the other hand, differentiating equation (9.6.1) with respect to $t$, one can obtain a linear parabolic differential equation for $v^+ = \partial_t u^+$. Since $v^+$ is zero at $t = -1$ and nonnegative on $\partial\Omega \times (-1, \infty)$, again by the monotonicity principle we have $\partial_t u^+ \ge 0$. Since the lateral boundary data for $u^+$ are constant for $t > 0$, this solution to the parabolic equation (9.6.1) is stabilizing to the solution $u^{+0}$ of the elliptic equation (9.6.8) with Dirichlet data $g^0 = G$. Due to monotonicity of $u^+$ with respect to $t$, we have $u^+(x,t) \le u^+(x)$. So $u(x,t) \le u^+(x)$, and any solution of our parabolic equation is majorized by a solution to the elliptic equation, so the sup over bounded solutions of parabolic equations is not greater than the sup over solutions of the steady-state elliptic equation. A similar argument works for inf. Therefore, $E_p \subset E$.

To prove the uniqueness of $a_0(x,u)$ we need the first three terms of the asymptotic expansion (9.6.11). Fix any $g^0, g^1, g^2 \in C^2(\partial\Omega)$. Since $c$ is known (on $E$), we know the solution $u^0$ to the Dirichlet problem (9.6.8). We therefore know a solution $u^1$ to the linear Dirichlet problem (9.6.9). The asymptotics (9.6.11) implies that $t^2\|\partial_\nu w(,t)\|_{(-1/2)}(\partial\Omega) \to 0$ as $t$ goes to $+\infty$. Since we are given $\partial_\nu u(,t)$ on $\partial\Omega$, multiplying (9.6.11) by $t$ and passing to the limit as $t \to +\infty$, we will find $\partial_\nu u^2$ on $\partial\Omega$. Multiplying both parts of equation (9.6.10) by an $H_{(2)}(\Omega)$- solution $v^2$ to the linear homogeneous equation

$$-\Delta v^2 + \partial_u c(x,u^0)v^2 = 0 \quad \text{in } \Omega \tag{9.6.12}$$

and using Green's formula, we obtain

$$\int_{\partial\Omega} (-(\partial_\nu u^2)v^2 + u^2\partial_\nu v^2) = \int_\Omega (-\frac{1}{2}\partial_u^2 c(x,u^0)(u^1)^2v^2 + a_0(x,u^0)u^1v^2).$$

For any given $g^0, g^1, g^2$, and $v^2$ the left side of this equality is known, as well as $c(x,u)$, $u^1$. Hence, for all solutions $u^1$ and $v^2$ to the linear Schrödinger equations (9.6.9) and (9.6.12) we are given the integrals

$$\int_\Omega a_0(x,u^0(x))u^1(x)v^2(x)dx.$$

By Corollary 5.3.5 these integrals uniquely determine $a_0(x,u^0(x))$ for all $x \in \Omega, u^0$, provided that $n \ge 3$. To be convinced that in fact $c$ is uniquely determined on $E$ is uniquely determined on $E$, it suffices to show that it is uniquely determined at any interior point $(x, u)$ of $E$. Let $u^0(x;\theta)$ be the solution to the (known) elliptic equation (9.6.8) with constant Dirichlet data $g^0 = \theta$. As already observed, the maximum (or monotonicity) principle implies that for any given $x$ the function $u^0(x;\theta)$ is increasing with respect to $\theta$. According to the definition of $E$ and to the previous remarks about monotonicity, there is a unique $\theta(x;u)$ such that $u^0(x;\theta) = u$. Since $c$ is known, so is the function $\theta(x;u)$. Finally, $a_0(x, u) = a_0(x, u^0(x;\theta(x, u)))$.

The proof is complete. □

For inverse problems in nonlinear equations with one measurement we refer to the paper of Pilant and Rundell [PiR1].

## 9.7 Interior sources

All the results of Chapter 9 presented so far about identification of coefficients have been obtained when the source term $f = 0$. This is the case in most important applications. With the exception of the final overdetermination, such inverse problems are quite unstable, and this is a serious obstacle for practical use. Lowe and Rundell [LowR] observed that when one can use the boundary data from many interior sources, then it is possible to solve inverse problems in a simple and stable way.

We consider a solution $u$ to the problem (9.0.1), (9.0.2), (9.0.3) with $a^0 = 1, a = 1, b = 0$, and unknown

(9.7.1) $$c \in C^\lambda(\overline{Q}).$$

We assume that the Dirichlet boundary data $g_0 = 0$ and that the source term $f$ is the Dirac delta function $\delta(-y, -s)$ with the pole at a point $(y, s)$. We denote the solution of this parabolic boundary value problem by $u(x, t; y, s)$. It is Green's function of the first boundary value problem for our equation.

Let $\gamma$ be any open part of $\partial\Omega$. Let us fix a function $g_0^*$ satisfying the following conditions:

$$0 \le g_0^* \text{ on } \gamma \times (0, T),\ g^* \text{ is not identically zero on } \gamma \times (s^0, T),$$
$$g_0^* = 0 \text{ on } (\partial\Omega\backslash\gamma) \times (0, T),\ g_0^* \in C^2(\partial\Omega \times [0, T]),$$
(9.7.2) $$g_0^* = \partial_t g_0^* = 0 \text{ on } \partial\Omega \times \{T\}.$$

**Theorem 9.7.1.** *The weighted flux integral*

$$w(y, s) = \int_{\Gamma\times(0,T)} g_0^* g_1(\ ; y, s)$$

*given for all $(y, s)$ in a neighborhood $V$ of a point $(y^0, s^0) \in Q$ uniquely (and in a stable way) determines $c(y^0, s^0)$.*

PROOF. Let $v$ be a solution to the backward heat equation

(9.7.3) $$\partial_t v + \Delta v = 0 \quad \text{in } Q$$

with initial and lateral boundary conditions

(9.7.4) $$v = 0 \text{ on } \Omega \times \{T\}, \qquad v = g_0^* \text{ on } \partial\Omega \times (0, t).$$

According to the definition of a generalized solution to our parabolic boundary value problem with the test function $v$, we have

$$\int_Q u(-\partial_t v - \Delta v + cv) - \int_{\gamma\times(0,T)} g_0^* \partial_\nu u = v(y, s).$$

Using equation (9.7.3) as well, we conclude that the function

$$U(y, s) = \int_Q cu(\ ; y, s)v = w(y, s) + v(y, s)$$

is given when $(y, s) \in V$. Since $u$ is Green's function for our parabolic equation, we have $-\partial_s U - \Delta_y U + cU = cv$, and using again equation (9.7.3), we conclude that

$$-\partial_s w - \Delta_y w + cw = 0 \text{ in } Q.$$

The function $w = U - v$ is 0 on $\Omega \times \{T\}$ and $-g_0^*$ on $\partial\Omega \times (0, T)$. By using conditions (9.7.2) and the positivity principle for parabolic equations, we conclude that $w < 0$ in $V$. Hence we can divide by $w$ in $V$ to obtain

(9.7.5) $$c(y, s) = (\partial_s w + \Delta w)/w(y, s), \quad (y, s) \in V.$$

The proof is complete. □

The reconstruction formula (9.7.5) gives a stable solution to the inverse problem.

We observe that in [LowR] the authors were looking for $c = c(x)$ in the one-dimensional case, and they used a minimal overdetermination over $\Gamma \times \{T\}$. They have been able to prove uniqueness under several very restrictive assumptions. We use more information, consider the multidimensional case, and impose minimal conditions on $c$.

Similarly, one can identify a leading coefficient. Consider the problem (9.0.1), (9.0.2), (9.0.3) with $a = 1$, $b = 0$, $c = 0$, unknown positive

$$a_0, \partial_t a_0 \in C^\lambda(\overline{Q}),$$

boundary data $g_0 = 0$, zero initial data, and the source term $f = \delta(-y, -s)$. Let $u(; y, s)$ be a solution of this problem. As above, let us introduce a function $g_0^{**}$ satisfying the conditions $0 \le \partial_t g_0^{**}$ on $\gamma \times (0, T)$; $g_0^{**}$ is not identically equal to zero on $\gamma \times (s^0, T)$; $g_0^{**} = 0$ on any $(\partial\Omega\backslash\Gamma) \times (0, T)$; $g_0^{**}, \partial_t g_0^{**} \in C^2(\partial\Omega \times [0, T])$; $g_0^{**} = \partial_t g_0^{**} = \partial_t^2 g_0^{**} = 0$ on $\partial\Omega \times \{T\}$.

**Exercise 9.7.2.** Show that if we introduce the weighted flux

$$w_1(s,t) = \int_{\gamma\times(0,T)} g_0^{**}\partial_\nu u(\,;s,t),$$

then

$$a_0(s,t) = -\Delta_y w/\partial_s w(s,t), \quad (y,s)\in V.$$

{*Hint*: Modify the proof of Theorem 9.7.1 using a function $v$ that solves the same backward heat equation with Dirichlet data $g_0^{**}$ instead of $g_0^{*}$. To show that $\partial_s w < 0$, differentiate the equation $-a_0\partial_s w - \Delta w = 0$ and the lateral boundary condition with respect to $t$ and make use of the positivity principle.}

## 9.8 Open problems

Here we mention some unsolved questions of significant theoretical and applied importance.

**Problem 9.1** (Two additional measurements at the final moment of time). It is true that two sets of boundary data $g_0, u_T$ (under some monotonicity conditions on $g_0$) uniquely determine two coefficients $a_0(x), c(x)$ in the inverse problem discussed in section 9.1?

A natural approach is to choose two different lateral boundary Dirichlet data $g_{01}, g_{02}$ such that $g_{02}$ is increasing faster then $g_{01}$ (e.g., $g_{01}(x.t) = e^t - 1$, $g_{02}(x,t) = e^{\lambda t} - 1$ with large $\lambda$), subtract two different equations as in the proof of Theorem 9.1.2, and try to eliminate one of the differences of unknown coefficients.

**Problem 9.2** (Global uniqueness for coefficients in the case of zero initial data and single measurements). Does the single set of lateral data $g_0, g_1$ with nonzero (monotone with respect to $t$) $g_0$ uniquely determine the coefficient $c = c(x)$ of the parabolic problem (9.0.1)–(9.0.3) with $a_0 = 1, a = 1, b = 0, f = 0$?

We emphasize that this is a problem for an arbitrary $g_0$, e.g., for $g_0(t) = t$. By using analyticity of solutions of $t$-independent problems with respect to $t$, it was observed by J. Gottlieb and T. Seidman that for some particular (and quite artificial) choice of

$$g_0(x,t) = \Sigma 2^{-k}\phi_k(t) g(x;k)$$

one can actually determine the Neumann data for all Dirichlet data $g(;k)$, recovering from one single boundary measurement the whole lateral Dirichlet-to-Neumann map when $\{g(;k)\}$ is a complete family of (smooth) functions on $\partial\Omega$. Here $\phi_k(t)\in C^\infty(\mathbb{R})$ is 0 when $0<t<1-2^{-k}$, is 1 when $1-2^{-k-1}-2^{-k-2}<t<1$, and otherwise $0<\phi_k(t)<1$. Responses ($\partial_\nu u_k$ on the lateral boundary) from all terms of the series can be separated by using $t$-analyticity of solutions of parabolic problems with $t$-independent data and coefficients and the special choice of the lateral boundary data $g_0$.

**Problem 9.3** (Global uniqueness of the $t$-dependent principal part). Does the lateral Dirichlet-to-Neumann map uniquely determine the coefficient $a_0 = a_0(x, t)$ in equation (9.0.1) with $a = 1, b = 0, c = 0, f = 0$?

Uniqueness is not known even when one is given the complete (not only lateral) Dirichlet-to-Neumann map $(g_0, u_0) \to (g_1, u_T)$. This is one of the simplest questions about uniqueness of the anisotropic partial differential equation. At present, the methods of complex geometrical optics described in Section 5.3 can give uniqueness only if $a_0$ is close to a constant. Of course, using analyticity arguments one derive generic uniqueness results similar to Theorem 5.5.3.

**Problem 9.4** (Restricted data for the domain problem). In the situation of Theorem 9.4.1 is it sufficient for uniqueness of $D$ with connected $\Omega \backslash \overline{D}$ in $a = 1 + \chi(D)$ to prescribe $\partial_\nu u$ only on $\partial\Omega \times \{T\}$?

**Problem 9.5** (Reduction of overdeterminacy in Theorem 9.5.2). Is uniqueness of the time-dependent domain valid in the case of a (special) single lateral boundary measurement?

Apparently, we required too much data in Theorem 9.5.2, but at present we have no idea how to get uniqueness with less data.

**Problem 9.6.** Does the lateral Dirichlet-to-Neumann map for the equation $\partial_t u - \Delta u + c(x, u) = 0$ in $\Omega \times (0, T)$ uniquely determine $c(x, u)$ when $T$ is finite?

This problem is about recovery of smooth $c$ that are not necessarily analytic with respect to $u$. But it has not been solved even for time-analytic coefficients, when one can try to mimic the proof of Theorem 9.5.1. This problem is quite interesting for systems , in particular in particualr in chemical kinetics, when it describes interaction term in mathematical models of chemical reactions.

# 10

# Some Numerical Methods

In this chapter we will briefly review some popular numerical methods widely used in practice. Of course it is not a comprehensive collection. We will demonstrate certain methods that are simple and widely used or, in our opinion, interesting and promising both theoretically and numerically. We observe that most of these methods have not been justified and in some cases even not rigorously tested numerically.

Due to the ill-posedness and nonlinearity of a typical inverse problem, one has two difficulties to overcome. In many practical numerical procedures nonlinearity is removed by replacing the original problem by its linearization around constant coefficients. Then a solution of the direct problem and therefore of the inverse one is dramatically simplified; but still, due to ill-posedness, the inverse problem is not easy, and its satisfactory numerical solution is possible mostly in the one- and two-dimensional cases. In Section 10.1 we take as an example the inverse conductivity and the inverse diffusion problems, treat their linearized versions, and briefly discuss some nonlinear methods. When the ill-posed problem admits Hölder stability estimates (as with the backward heat equation or with the Cauchy problem with the data on pseudo-convex surfaces, or related coefficients problems for equations), there is a modification of the general regularization method that generates efficient numerics. We will present this approach in Section 10.2 together with the conjugate gradient method applied to the problem of numerical continuation of the acoustic field with applications in nearfield acoustic holography. A nonlinear identification of coefficients can be reduced to a nonconvex minimization problem, and there have been several attempts to find a minimizer numerically. One serious difficulty is connected with nonconvexity, which generates several possible local (but not global) minima, so one can try to replace minimization of a nonconvex functional by minimization of a convex one. This replacement is called *relaxation* and is intensively studied in the calculus of variations and in the theory of optimal control. In section 10.3 we present one adaption of these methods to the inverse conductivity problem as well as another approach to convexification based on Carleman estimates in Section 10.3. Invariant embedding methods have been applied quite successfully applied to one-dimensional inverse hyperbolic problems. Recently, they have been adjusted to the inverse conductivity problem. We will discuss this topic in Section 10.4. In section 10.5 we collect describe various methods based on the range test. Indeed, growing computational power made possible a complete spectral analysis (singular value decomposition) of operators arising in important inverse problems. Range tests are deciding whether a point is contained in a support of a inhomogeneity by checking if a some particular function is contained in the range of the operator determined by the data of the inverse problem. Typically they can detect a domain or point of a support of some source function from data of an overdetermined inverse problem, and even if the original inverse problem is nonlinear, range tests are typically implemented by solving linear problems. Finally, in concluding Section 10.6, we report on some results on the discretized inverse conductivity problem, which can be also interpreted as reconstruction of resistances in a resistor network from exterior measurements.

## 10.1 Linearization

In Section 4.5 we justified in a certain way linearization of inverse coefficients problems, in particular, of the inverse conductivity problem. The linearized inverse conductivity problem consists in finding the "small" perturbation $f = a_\delta$ of the constant conductivity coefficient $a = 1$ entering the boundary value problem

$$\begin{aligned} -\Delta v &= \operatorname{div}(f \nabla u_0) \text{ in } \Omega, \\ v &= 0 \text{ on } \partial\Omega \end{aligned} \tag{10.1.1}$$

from the additional Neumann data

$$\partial_\nu v = g_1 \text{ on } \partial\Omega \tag{10.1.2}$$

given for any harmonic function $u_0$ with Dirichlet boundary data $g_0$. By using Green's formula when $f$ is $C^1$-smooth or the definition of the weak solution in the general case, we obtain

$$\int_\Omega f \nabla u_0 \cdot \nabla w = \int_{\partial\Omega} g_1 w, \tag{10.1.3}$$

where $w$ is any harmonic function (in $\mathbb{R}^n$). Since $w$ and $g_1$ are known functions, we are given the integral on the left side. This approach was used by Calderon [C], who let $u_0(x) = 2|\xi|^{-1} \exp(-\frac{1}{2}(i\xi + \xi^\perp) \cdot x)$, $w(x) = |\xi|^{-1} \exp(-\frac{1}{2}(i\xi - \xi^\perp) \cdot x)$, to obtain the Fourier transformation

$$\int_\Omega f(x) e^{-i\xi \cdot x} dx,$$

which uniquely (and in a stable way) recovers $f$. Here $\xi^\perp$ denotes a vector perpendicular to $\xi \in \mathbb{R}^n$ of the same length as $\xi$. The needed boundary data $g(x)$ are $u_0(x; \xi)$, $x \in \partial\Omega$. Instability in this procedure comes from the exponentially growing factors $\exp(-\xi^\perp \cdot x)$. A similar approach was utilized by David Isaacson and his group at the Rensselaer Polytechnic Institute for a practical implementation of electrical impedance tomography in $\mathbb{R}^2$ and $\mathbb{R}^3$. In particular, they considered the unit disk $\Omega$ in $\mathbb{R}^2$ and cylinder in $\mathbb{R}^3$, which simulate the human body, and prescribed sines and cosines as boundary data for $u_0$, taking frequencies $|\xi| \leq 100$. The problem was to find the conductivity of the human body from all possible boundary electrical measurements. The results of reconstruction in real time gave a recognizable conductivity. The resolution of this reconstruction method cannot compete with, say, classical tomography, which is quite understandable and natural due to the severe ill-posedness of the inverse conductivity problem. Eventually, the Rensselaer group employed a more sophisticated model by using more realistic boundary conditions but the same linearization approach. However, the reconstruction results still do not completely satisfy practitioners, and it is a challenging and important question whether by improving the model, precision of measurements, and most of all their mathematical processing, one would be able to design a new and powerful medical diagnostic tool. The review of these results is given in [CheIN].

Now we will describe another linearization method based on the constructive approach, which reduces the inverse conductivity problem to a linear integral equation with Riesz kernel and which is close to the original method of Barber and Brown, inventors of electrical impedance tomography. Indeed, let $u_0(y) = K(x, y)$ and $w(y) = K(x, y)$, where $K$ is a classical fundamental solution to the Laplace equation, $x \in \Omega^*$, and $y \in \Omega$. Here $\Omega^*$ is a domain in $\mathbb{R}^n$ with $\overline{\Omega}^* \cap \overline{\Omega} = \emptyset$. Since $u_0, w$ are given, as well as functions $g_1$ corresponding to the boundary data $g_0 = u_0$ on $\partial\Omega$, from (10.1.3) we know that

$$F(x) = \int_\Omega |x - y|^{2(1-n)} f(y)dy, \quad x \in \Omega^*, \tag{10.1.4}$$

where $F$ is the right side of (10.1.3) divided by $c(n)$, $c(2) = (2\pi)^{-2}$, and $c(3) = (4\pi)^{-2}$. The equality (10.1.4) is an integral equation with respect to the unknown function $f$. We have already indicated in Section 2.2 that this equation is of the first kind, that its solution $f \in L_2(\Omega)$ is unique, and that there are conditional logarithmic stability estimates of its solution. In the paper of Isakov and Sever [IsS] there are results of numerical experiments with this equation when $\Omega$ is the unit disk in $\mathbb{R}^2$ and $\Omega^*$ is the annulus $\{y : 2 < |y| < 3\}$. We have used $10 \times 10$ and $30 \times 30$ grids in polar coordinates in both domains and discretized the integrals replacing them by the mean value theorem. Minor errors (of relative magnitude 1% in the uniform norm) completely destroyed the computations, so we regularized equation (10.1.4) by the following one:

$$\begin{aligned} &\alpha f(x;\alpha) + \int_{\Omega\times\Omega^*} |x - v|^{-2}|v - y|^{-2} f(y;\alpha)dv\, dy \\ &= \int_{\Omega^*} |x - v|^{-2} F(v)dv, \end{aligned} \tag{10.1.4$_\alpha$}$$

where $x \in \Omega$. Equation (10.1.4$_\alpha$) is the particular form of the regularization (2.3.4) when $x(\alpha) = f(;\alpha)$, $x^0 = 0$, and the operator $Af(x)$ is defined by the right side of equation (10.1.4). Then the discretized equation (a linear system with from 100 to 1000 unknowns) was solved using the conjugate gradient method. This simple regularization with $\alpha = 10^{-5}$ guaranteed a very good reconstruction of simple polynomial functions in $\Omega$, like $f(y) = y_1^2 - y_2^2$. For discontinuous $f = k\chi_D$ there are also satisfactory results when $D$ is a disk or radius $\frac{1}{2}$ or two disks of radius $\frac{1}{5}$, but a disk of radius $\frac{1}{10}$ was unrecognizable. Furthermore, discontinuous $f$ require $\alpha = 10^{-10}$.

Now we will discuss a linearized inverse diffusion problem. We would like to find the diffusion coefficient $a$ of the parabolic initial boundary value problem

$$\begin{aligned} \partial_t u - \operatorname{div}(a\nabla u) = 0 \quad & \text{in } Q = \Omega \times (0, T), \\ u = 0 \quad & \text{on } \Omega \times \{0\}, \\ u = g_0 \quad & \text{on } \partial\Omega \times (0, T). \end{aligned} \tag{10.1.5}$$

The additional data are given by the Neumann boundary condition

$$a\partial_\nu u = g_1 \quad \text{on } \partial\Omega \times (0, T). \tag{10.1.6}$$

We will assume that the complete lateral Dirichlet-to-Neumann map $g_0 \to g_1$ is given. As above, we will linearize the inverse problem around $a = 1$, letting $a = 1 + f$, with $f = f(x)$ either uniformly small in $\Omega$ or of the form $k\chi_D$, where $k$ is a $C^2(D)$-bounded function and $D$ is a domain of small volume with bounded perimeter. To adjust the argument of Sections 4.5 to parabolic equations, it suffices to use Theorem 9.1 instead of Theorem 4.1. We arrive at the linearized inverse diffusion problem of finding $f$ entering the parabolic boundary value problem

$$\begin{aligned} \partial_t v - \Delta v = \operatorname{div}(f\nabla u_0) \quad & \text{on } Q, \\ v = 0 \quad & \text{on } \Omega \times \{0\}, \\ v = 0 \quad & \text{on } \partial\Omega \times (0, T) \end{aligned} \tag{10.1.7}$$

from the additional Neumann data

$$a\partial_\nu v = g_1 \text{ on } \partial\Omega \times (0, T) \tag{10.1.8}$$

given for all solutions $u_0$ to the heat equation in $Q$ that are zero on $\Omega \times \{0\}$. As for the conductivity equation, using the definition of a generalized solution (9.0.4) or Green's formula for regular solutions, we obtain the identity

$$\int_Q f\nabla u_0 \cdot \nabla w = \int_{\partial\Omega\times(0,T)} g_1 w \tag{10.1.9}$$

for any solution $w \in H_{2,1;2}(Q)$ to the backward heat equation in $Q$ that is zero on $\partial\Omega \times \{T\}$. Since the right side of this equality is given from the data of the inverse problem with $g_0 = u_0$, we are given the left side. Choosing

$$\begin{aligned} u_0(y, s) &= (4\pi s)^{-n/2} \exp(-|x - y|^2/(4s)), \\ w(y, s) &= (4\pi(T - s))^{-n/2} \exp(-|x - y|^2/(4(T - s)) \end{aligned}$$

and calculating gradients, we arrive at the integral equation

$$F(x;T) = \int_{\Omega\times(0,T)} k(x-y)f(y)dy,\ x \in \Omega^*, \tag{10.1.10}$$

where

$$k(x) = -|x|^2 \int_0^T (\tau(T-\tau))^{-n/2-1} \exp(-|x|^2T/(4\tau(T-\tau))d\tau$$

and the data $F$ are obtained by multiplying the (known) right side of (10.1.9) by $4(4\pi)^n$. Now we will show that a solution of this equation is unique.

**Lemma 10.1.1.** *A solution $f \in L_2(\Omega)$ to the integral equation* (10.1.10) *is unique.*

PROOF. It suffices to assume that $F = 0$ on $\Omega^*$ and to show that $f = 0$. Assume the contrary. Using that the Fourier transformation of the convolution is a product of Fourier transformations, we obtain $\hat{F}(\xi) = \hat{k}(\xi)\hat{f}(\xi)$. Since $F = 0$ on $\Omega^*$ and is real-analytic outside $\overline{\Omega}$, we conclude that $F = 0$ outside $\Omega$. Since $F, f$ have compact supports, by the Paley-Wiener theorem $\hat{F}, \hat{f}$ are entire analytic functions of order 1, so $|\hat{f}(\zeta)| \leq C\exp(C|\zeta|)$ when $\zeta \in \mathbb{C}$, and the same inequality holds for $f(\zeta)$. Since $f$ is not identically zero, $f(\xi_0) \neq 0$ for some $\xi_0 \in \mathbb{R}^n$. Hence the entire analytic function of one complex variable $\phi(z)$ defined as $f(z\xi_0)$ is of order 1 and not identically zero. By a theorem of Littlewood there is a constant $C$ and a sequence of positive $r_j \to \infty$ such that $|\phi(ir_j)| > \exp(-Cr_j)$. Using the above bound for $\hat{F}$, we conclude that

$$|\hat{k}(ir_j\xi_0)| \leq |\hat{F}(ir_j\xi_0)|/|\hat{f}(ir_j\xi_0)| \leq C\exp(Cr_j).$$

On the other hand, by the result of Exercise 10.1.2 we have

$$\begin{aligned}
|\hat{k}(i\xi)| &= c(n)|\int_0^T (2\tau(T-\tau)/T|\xi|^2 + n)\exp(|\xi|^2\tau(T-\tau)/T)d\tau| \\
&\geq nc(n)\int_0^T \exp(|\zeta|^2\tau(T-\tau)/T)d\tau \\
&\leq nc(n)\int_{T/4}^{T/2} \exp(|\zeta|^2 3T/16)d\tau \\
&\geq nc_nT/4\exp(3T/16|\zeta|^2),
\end{aligned}$$

so we can claim that $|\hat{k}(\zeta)| > C^{-1}\exp(|\zeta|^2/C)$ when $\zeta = i\xi$. This contradicts the previous bound on $\hat{k}$ when $r_j$ is large.

This contradiction shows that the initial assumption was wrong. The proof is complete. □

**Exercise 10.1.2.** Show that the Fourier transform $\hat{k}(\xi) = 2^{-1-n}T^{n/2-1}\pi^{-n/2}$ $\int_0^T (2\tau(T-\tau)/T|\xi|^2 - n)\exp(-|\xi|^2\tau(T-\tau)/T)d\tau$

{*Hint*: Use the Fourier transform of the function $\exp(-\gamma|x|^2/2)$ is $(2\pi)^{n/2}\gamma^{-n/2}\exp(-|\xi|^2/(2\gamma))$ and that $(|x|^2\hat{u}(x)) = -\Delta_\xi\hat{u}(\xi)$.}

So far, there is no stability estimate for the integral equation (10.1.10), but one expects a logarithmic estimate as for equation (10.1.4).

We will report on some numerical results obtained in the paper of Elayyan and Isakov [ElI2]. As in the elliptic case, one can use the Tikhonov regularization, replacing (10.1.10) by the equation

$$\alpha f_\alpha + A_T^* A_T f_\alpha = F^*, \tag{10.1.11}$$

where $A_T$ is given by the right side of equation (10.1.10), and correspondingly,

$$A_T^* A_T f(x) = (16(4\pi)^{2n}) \int_{\Omega^* \times \Omega} f(y)|y-w|^2|x-w|^2$$
$$\times K(x-w,T)K(y-w,T)dydw,$$

with

$$K(x-w,T) = \int_0^T (\tau(T-\tau))^{-n/2-1} \exp(-|x-w|^2 T/(r\tau(T-\tau))d\tau$$

and $F^* = A_T^* F$. A numerical solution is more difficult because one has to compute kernels that in the elliptic case were given by simple formulae.

In [ElI2] we considered $\Omega = (0,2)$ and $\Omega^* = (3,5)$ when $n = 1$ and $\Omega = (0,2) \times (0,2)$ and $\Omega^* = (3,5) \times (0,2)$ when $n = 2$ and discretizing integrals by the trapezoid method that have been discretized by $N$ and $N \times N$ grids with $N = 30$. In the one-dimensional case a reconstruction of the function $f(x) = \sin \pi x_1$ was very good with $\alpha = 10^{-6}$, $T = 4$, and it survived adding 1% relative noise. In the two-dimensional case the reconstruction of the same function was even better with $\alpha = 10^{-10}$, though much more time-consuming. Also, it was possible to recover a simple discontinuity: $f(x) = 1$ when $1 < x_1 < 2$, $0 < x_2 < 2$, and zero elsewhere.

We observe that the Tikhonov regularization, while simple in implementation and quite general, does not utilize some possible a-priori features of functions $f$. For example, when $f$ is smooth, one can use other regularizations, say, adding to the residue the norm in $H_{(k)}(\Omega)$. Another way to improve the computations is to utilize symmetry of kernels that depend on the difference of arguments and generate a Töplitz structure of discretized equations. A substantial improvement has already been made by Chan with coauthors [Cha], who made use of preconditioners. In particular, they took $k = 1$, replacing in (10.1.11) $\alpha f(;\alpha)$ by $-\alpha\Delta f(;\alpha)$.

Another approach was suggested by Berenstein and Tarabusi [BerT], who considered an integral equation similar to (10.1.4) and decomposed its solution into a (stable) inversion problem of integral geometry over spheres with weights and a one-dimensional convolution equation. This approach, called a back-projection algorithm, was studied (in particular, numerically) by Santosa and Vogelius [SaV1].

Several numerical methods are discussed in the review paper of Borcea [Bor], in particular use of constraint minimization, high-contrast conductivites, and probabilistic methods. In many applications, including electrical impedance tomography, the unknown function $f$ is discontinuous, more precisely, $f = k\chi_D$, where $D$ is the union of finitely many bounded Lipschitz domains, with $k$ constant on any such domain and unknown, and one has good a priori information about maximal and minimal values of $k$ and about the perimeter of $D$. Then regularization by higher Sobolev norms does not make much sense, and after the work of Osher and Setjan [OshS], many researchers started to use as a regularizing addition the functional of total variation

$$TV(f) = \|\nabla f\|_1(\Omega).$$

This functional, while very natural, generates a nonlinear Euler equation for a minimum point, replacing, for example, $\alpha f(;\alpha)$ in the equation (10.1.4) with $-\alpha\,\mathrm{div}(|\nabla f|^{-1}\nabla f)$. The computational effort is accordingly greater, but the quality of the reconstruction of the discontinuous $f$ improves. For an analysis of the numerical solution when regularizing by the total variation functional, we refer to the paper of Dobson and Santosa [DoS]. A particular and very effective method of recovery of discontinuuty surface of conductivity based on single layer representation is given by Kwon, Seo, and Yoon [KwSY].

## 10.2 Variational regularization of the Cauchy problem

The numerical method of variational regularization can be considered as a concretization of regularization of differential problems. It was developed recently by Klibanov and his collaborators, and we describe it using the results of the paper by Klibanov and Rakesh [KlR], where they discussed the case $a_0 = 1$ and $\gamma = \partial\Omega$.

We consider the hyperbolic equation

(10.2.1) $$(a_0^2\partial_t^2 u + Au) = f \quad \text{in } Q = \Omega \times (-T, T),\, u \in H_{(2)}(Q),$$

with Cauchy data

(10.2.2) $$u = g_0,\, \partial_\nu u = g_1 \quad \text{on } \Gamma = \gamma \times (-T, T).$$

We assume that $a_0 = a_0(x) \in C^1(\overline{\Omega})$ and is positive on $\overline{\Omega}$ and that $A$ is the elliptic operator $-\Delta + c$, $c \in L_\infty(Q)$. Generally, this problem is not well-posed, and even when one has Lipschitz stability (as in Theorem 3.4.5), it is overdetermined, so its numerical solution needs some version of the least-squares method. When solving the Cauchy problem (10.2.1)–(10.2.2), we can assume that $g_0 = 0$, $g_1 = 0$. Indeed, we can always achieve this by extending the Cauchy data onto $\Omega$ as a function $u^*$ and subtracting $u^*$ from $u$, obtaining Cauchy data for the difference zero. The extension operator is continuous from $H_{(3/2)}(\gamma \times (-T, T)) \times H_{(1/2)}(\gamma \times (-T, T))$ and can be explicitly written down for simple domains $\Omega$. Later on, we assume that $g_0 = g_1 = 0$.

The general regularization scheme (2.3.3) is quite applicable here. Specifically, we can solve the following (regularized) minimization problem:

(10.2.3) $$\min(\|a_0^2\partial_t^2 v + Av - f\|_2^2(Q) + \alpha\|v\|_{\bullet(2)}^2(Q)),\, v \in \mathring{H}_{(2)}(Q;\Gamma),$$

where $\|v\|_{\bullet(2)}(Q)$ is one of the equivalent norms in $H_{(2)}(Q)$ defined as

$$\left(\sum_{1\leq j\leq n+1}\|\partial_j^2 v\|_2^2(Q)+\|v\|_2^2(Q)\right)^{1/2}$$

and $\mathring{H}_{(2)}(Q;\Gamma)$ is the subspace of $H_{(2)}(Q)$ formed of functions $v$ with $v=\partial_\nu v=0$ on $\Gamma$. Due to coercivity and convexity of the regularized functional (10.2.3) on the Hilbert space $\mathring{H}_{(2)}(Q;\Gamma)$, the minimum point $u(;\alpha)$ in this space exists and is unique by known basic results of convex analysis; see the book of Ekeland and Temam [ET]. To describe the Euler equation for the minimum point $u(;\alpha)$ we need a new scalar product $[,](Q)$ in $H_{(2)}(Q)$, defined as follows:

(10.2.4)

$$[u,v](Q)=\int_Q\left(a_0^2\partial_t^2 u+Au\right)\left(a_0^2\partial_t^2 v+Av\right)+\alpha\sum_{1\leq j\leq n+1}\partial_j^2 u\partial_j^2 v+\alpha uv.$$

Then a minimum point of (10.2.3) is a minimum point of the quadratic functional

$$[v,v](Q)-2\left(f,a_0^2\partial_t^2 v+Av\right)_2(Q),$$

and by standard technique we have the variational Euler equation

$$[u(;\alpha),v](Q)=\left(f,a_0^2\partial_t^2 v+Av\right)_2(Q),\ v\in\mathring{H}_{(2)}(Q;\Gamma), \tag{10.2.5}$$

for the minimum point $u(;\alpha)$.

**Lemma 10.2.1.** *We have*

$$\begin{aligned}&\|u-u(\,;\alpha)\|_{\bullet(2)}(Q)\leq\|u\|_{\bullet(2)}(Q),\\ &\|(a_0^2\partial_t^2+A)(u-u(\,;\alpha))\|_2(Q)\leq 2^{-1/2}\alpha^{1/2}\|u\|_{\bullet_{(2)}}(Q).\end{aligned}\tag{10.2.6}$$

PROOF. Since $u$ solves equation (10.2.1), from the definition (10.2.4) of the scalar product $[,]$ we have

$$[u,v](Q)=\left(f,\left(a_0^2\partial_t^2 v+Av\right)\right)_2(Q)+\alpha(u,v)_{\bullet(2)}(Q).$$

Subtracting this equality from the definition (10.2.5) of the weak solution $u(\,;\alpha)$, we obtain

$$[u(\,;\alpha)-u,v](Q)=-\alpha(u,v)_{\bullet 2}(Q),\quad v\in\mathring{H}_{(2)}(Q;\Gamma).$$

Letting $v=u(\,;\alpha)-u$ and using the definition (10.2.4) of the scalar product $[,]$ gives

$$\begin{aligned}&\|(a_0^2\partial_t^2+A)(u(\,;\alpha)-u)\|_2^2(Q)+\alpha\|u(\,;\alpha)-u\|_{\bullet(2)}^2(Q)\\ &=-\alpha(u,u(\,;\alpha)-u)_{\bullet(2)}(Q)\leq\alpha\|u\|_{\bullet(2)(Q)}\|u(\,;\alpha)-u\|_{\bullet(2)}(Q)\\ &\leq\alpha/2\|u(\,;\alpha)-u\|_{\bullet(2)}^2(Q)+\alpha/2\|u\|_{\bullet(2)}^2(Q),\end{aligned}\tag{10.2.7}$$

where we used the Schwarz inequality for the scalar product $(,)_{\bullet(2)}$ and the elementary inequality $ab\leq\frac{1}{2}(a^2+b^2)$. Subtracting the first term of the last quantity in

(10.2.7) and dropping the first term of the first quantity we obtain the first inequality (10.2.6). Dropping the second term gives the second inequality (10.2.6).

The proof is complete. □

Lemma 10.2.1 does not guarantee that $u(\ ;\alpha)$ is convergent to $u$ in $L_2(Q)$ when $\alpha \to 0$ (and generally this is false, due to nonuniqueness in the Cauchy problem in $Q$). However, combining the results of Lemma 10.2.1 and of Theorems 3.4.1 and 3.4.5, one can obtain convergence in domains $Q_\varepsilon$ determined by a weight function in Carleman estimates for the domain $Q = \Omega \times (-T, T)$ in the cases (3.4.2), (3.4.3), or even in all of $Q$ under the conditions of Theorem 3.4.5.

**Corollary 10.2.2.** *If the domain $Q$ and the coefficient $a_0$ satisfy the conditions of Theorem 3.4.1, then*

(10.2.8) $$\|u(\ ;\alpha) - u\|_{(1)}(Q_\varepsilon) \le C\alpha^\lambda \|u\|_{(2)}(Q),$$

*where $\lambda \in (0, 1)$ depends on $\varepsilon > 0$ determining the domain $Q_\varepsilon$.*

*If the domain $Q$ and the coefficient $a_0$ satisfy the conditions of Theorem 3.4.5 and $\gamma = \partial\Omega$ then*

(10.2.9) $$\|u(;\alpha) - u\|_{(1)}(Q) \le C\alpha^{1/2} \|u\|_{(2)}(Q).$$

PROOF. To prove (10.2.8) we observe that the first bound (10.2.6) implies that $\|u(\ ;\alpha) - u\|_{(2)}(Q) \le C\|u\|_{(2)}(Q)$, and we combine the second bound (10.2.6) and the bound (3.4.7) of Theorem 3.4.1.

To prove (10.2.9) we observe that Theorem 3.4.5 implies the bound

$$\|w\|_{(1)}(Q) \le C\|(a_0^2\partial_t^2 + A)w\|_2(Q),$$

provided that $w \in H_{(2)}(Q)$ and $w = \partial_\nu w = 0$ on $\partial\Omega \times (-T, T)$. Let $w = u(\ ;\alpha) - u$ and use the second bound (10.2.6). □

In the paper [K1R] there are numerical results for this regularization in the case of the wave equation in $\mathbb{R}^2$. The authors found solutions $u(\ ;\alpha)$ of the regularized problem (after discretization) by a finite-element method. The numerical experiment agrees with the convergence estimate very well.

Similarly, one can consider the lateral Cauchy problem for parabolic equations

(10.2.10) $$a_0\partial_t u + Au = f \text{ in } Q, \quad u \in H_{2,1;2}(Q),$$

with the lateral boundary data

(10.2.11) $$u = g_0, \partial_\nu u = g_1 \quad \text{on } \Gamma \times (-T, T).$$

In this case we keep the assumptions about $a_0$ while considering a general elliptic operator $A$ of second order with time-dependent coefficients, $C^1(\overline{\Omega})$-principal coefficients, and other coefficients in $L_\infty(Q)$. As above, for a solution of this problem we can assume that $g_0 = g_1 = 0$.

In the case of parabolic equations, we define

$$\|v\|^2_{\bullet 2,1;2}(Q) = \|\partial_t v\|^2_2(Q) + \sum_{1\leq j\leq n} \|\partial_j^2 v\|^2_2(Q) + \|v\|^2_2(Q).$$

**Exercise 10.2.3.** Show that there is a unique minimizer $u(\ ;\alpha)$ of the problem

$$\min(\|a_0^2\partial_t v + Av\|^2_2(Q) + \alpha\|v\|^2_{\bullet 2,1;2}(Q))$$

over $v \in H_{2,1;2}(Q)$ with $v = \partial_\nu v = 0$ on $\gamma \times (-T, T)$. Show that

$$\|u - u(\ ;\alpha)\|_{\bullet 2,1;2}(Q) \leq \|u\|_{\bullet 2,1;2}(Q),$$

$$\|(a_0\partial_t v + Av)(u - u(\ ;\alpha)\|_2(Q) \leq 2^{-1/2}\alpha^{1/2}\|u\|_{\bullet 2,1;2}(Q) \tag{10.2.12}$$

and derive from Theorem 3.3.10 that for any domain $Q_\varepsilon$ with closure in $Q \cup (\gamma \times (-T, T))$ there are $C$ and $\lambda \in (0, 1)$ such that

$$\|u - u(\ ;\alpha)\|_{2,1;2}(Q_\varepsilon) \leq C\alpha^\lambda\|u\|_{2,1;2}(Q).$$

Also, one can solve in this manner the backward initial problem for equation (10.2.10) with final and lateral boundary data

$$u(, T) = u_T,\ \text{on } \Omega \times \{T\}, u = 0 \text{ on } \partial\Omega \times (0, T). \tag{10.2.13}$$

As above, we can use extension results and consider $u_T = 0$.

**Exercise 10.2.4.** Show that there is a unique solution $u(\ ;\alpha)$ of the minimization problem

$$\min(\|\partial_t v + Av - f\|^2_2(Q) + \alpha\|v\|^2_{\bullet 2,1;2}(Q))$$

over $v \in H_{2,1;2}(Q)$ with $v = 0$ on $\partial\Omega \times (0, T)$ and on $\Omega \times \{0\}$.

Further, show that

$$\|(u - u(\ ;\alpha))(t)\|_2(\Omega) \leq C(\alpha^\lambda + \alpha^{1/2})\|u\|_{2,1;2}(Q), \lambda = \lambda(t) \in (0, 1),$$

and that for $A = -\Delta + c$ one can choose $\lambda(t) = t/(2T)$.

{*Hint*: Again make use of the scheme of the proof of Lemma 10.2.1 to obtain the bounds (10.1.12). Then the difference $w = u - u(\ ;\alpha)$ will solve the parabolic equation (10.2.10) with the right side $f_1$ such that

$$\|f_1\|_2(Q) \leq C\alpha^{1/2}\|u\|_{2,1;2}(Q)$$

and will have zero boundary and final data. Use the solution $w_1$ of the initial boundary value problem for (10.2.10) with right side $f_1$ and with zero boundary and initial data and bound this solution using Theorem 9.1 and the trace theorem. to bound the solution of the homogeneous parabolic equation for $w - w_1$ use Theorem 3.1.3 and Example 3.1.6.}

So far, we have considered only linear problems or linearizations, while identification of coefficients creates at least quadratic nonlinearity. Some Newton-type algorithms make use of similar regularization and have been proved to be effective, but there is no general global approach to this problem.

Another interesting "alternating" iterative method has been suggested in applications and considered and justified in general framework in the paper of Kozlov and Maz'ya [KozM]. For the Laplace equation in a domain $\Omega$ with Cauchy data given on $\Gamma \subset \partial\Omega$ it consists in first solving the mixed boundary value problem with given data on $\Gamma$ and zero Neumann data on $\partial\Omega\backslash\Gamma$; then solving the mixed boundary value problem with given Neumann data on $\Gamma$ and the Dirichlet data on the remaining part, which are obtained at the previous step; and then iterating. This algorithm is convergent, and it would be interesting to develop its analogue for the problems of identification of coefficients.

As an example of efficient numerical solution of practically important problem we consider regularization of a linear integral equation arising in nearfield acoustical holography. This technique seeks for vibrations of a surface from the acoustical pressure generated by these vibrations. We will describe recent results of DeLillo, Isakov, Valdivia, and Wang who handled a problem from aircraft industry. The component $u$ of pressure of frequency $k$ satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \text{ in } \Omega.$$

Here $\Omega$ is a Lipschitz domain in $\mathbb{R}^3$ with connected $\mathbb{R}^3\backslash\overline{\Omega}$. Microphones are located on a surface $\Gamma_0$ inside (cabin) $\Omega$, so we are given

$$u = g_0 \text{ on } \Gamma_0.$$

One is looking for the so-called normal velocity

$$v = \partial_\nu u \text{ on } \Gamma = \partial\Omega.$$

It was shown in [DIVW] that any $u \in H_{(1)}(\Omega)$ admits unique representation by the single layer potential distributed over $\Gamma$ (disregard of Dirichlet or Neumann eigenvalues of the Laplacian in $\Omega$). So it was proposed in [DIVW] to solve for density $\phi$ from the integral equation

$$1/(4\pi)\int_\Gamma e^{ik|x-y|}/|x-y|\phi(y)d\Gamma(y) = u(x),\ \ x \in \Gamma_0 \tag{10.2.14}$$

and then to find $\partial_\nu u$ on $\Gamma$ from known jump relations for normal derivative of single layer potential. Uniqueness can be guaranteed if $\Gamma_0$ is a boundary of a domain of small volume. Under constraints similar to those in Theorem 3.3.1 one has logarithmic stability, but since distance from $\Gamma_0$ to $\Gamma$ is relatively small one can achieve high resolution.

The integral equation (10.2.14) was discretized in [DIVW] by using piecewise linear triangular boundary elements and the resulting linear algebraic system with $N$ unknowns was solved by the iterative conjugate gradient method where number $J$ of iterations plays the role of regularization parameter. The choice of $J$ is cricial for efficient numerics. When $\Omega$ is a cylinder of radius 1 with floor and flat endcaps modeling Cessna 650 fuselage, $k = 3$ (typical acoustic range), and $N = 1000$, we used $J = 30$ as suggested by generalized cross-validation. Relative $L_\infty$-error of

0.01 in data produced relative reconstruction error of 0.1. Our experience showed that use of integral equations is the most effective in numerical solution of ill-posed Cauchy problems of relatively large size. Of course, it presumes a simple analytic form of a fundamental solution. So far single layer method is the most competitive algorithm for nearfield acoustic holography which is suitable for any geometry of $\Omega$ and for exterior problems typical for automotive industry and naval applications.

## 10.3 Relaxation methods

A very serious difficulty in solving a coefficient-identification problem is due to the nonconvexity of the corresponding minimization problem. Naively, a minimum point of a (nonconvex) function coincides with a minimum point of the greatest convex function that is pointwise not greater than the original function. So for some time in the theory of variational methods one has considered replacing a nonconvex functional to be minimized by its convexization. This new convex minimization problem is called a *relaxation* of the original one. To be justified, the method needs some conditions on the original functional, and there is a useful discussion of available technique and results in the book of Ekeland and Temam [ET]. Before returning to inverse problems, we recall a few basic facts about relaxation.

Let $\Phi(x, v)$ be a (nonlinear) function of $v \in \mathbb{R}^m$. Its polar function $\Phi^*(x, v^*)$ is defined as

$$\Phi^*(x, v) = \sup(v \cdot v^* - \Phi(v)) \text{ over } v \in \mathbb{R}^m,$$

and the second polar $\Phi^{**}$ coincides with the pointwise supremum of all linear functions that are (pointwise) less than $\Phi$. We call the second polar function a $\Gamma$-regularization of $\Phi$.

While in principle, $\Gamma$-regularization gives the needed convex replacement of the original functional, its constructive calculation has been implemented only for special functionals $\Phi$. One can consult the book of Ekeland and Temam [ET] about the relaxation of the classical variational problem

$$\min \int_\Omega f(x, u(x), \nabla u(x))dx$$

over $u \in H_{1,p}(\Omega)$, $u - u_0 \in \mathring{H}_{1,p}(\Omega)$, which is to find

$$\min \int_\Omega f^{**}(x, u(x), \nabla u(x))dx$$

over the same convex set of functions. Here $f^*(, v) = \sup(v \cdot v^* - f(, v))$, where the arguments $x$, $u$ are dropped, and accordingly, one calculates $f^{**}$. Of course, one can assume that $f$ is a normal integrand with some growth at infinity with respect to $v$. However, an inverse problem has originally no classical variational form, and in any particular case it is not quite clear how to reduce it to this form. We will discuss one case in which this reduction can be done.

We will demonstrate the variational approach to the inverse conductivity problem due to Kohn and Vogelius [KoV3]. We are looking for the conductivity coefficient $a$ from the results of $N$ boundary measurements of voltages and currents at the boundary. The equation and boundary conditions are

$$\operatorname{div}(a\nabla u_j) = 0 \text{ in } \Omega,\ j = 1, \dots, N,$$
$$u_j = g_{0,j},\ a\partial_\nu u_j = g_{1,j} \text{ on } \partial\Omega. \tag{10.3.1}$$

A variational method is to find

$$\min K(u_1, \dots, u_N; v_1, \dots, v_N; a)$$

subject to the constraints

$$u_j = g_{0,j},\ v_j \cdot \nu = g_{1,j} \text{ on } \partial\Omega,\ u_j, v_j \in H_{(1)}(\Omega),$$
$$\operatorname{div} v_j = 0,\ C_1 \le a \le C_2 \text{ in } \Omega, \tag{10.3.2}$$

where the functional

$$K = \int_\Omega K(\nabla u_1, \dots, \nabla u_N; v_1, \dots, u_N; a) \tag{10.3.4}$$

with the integrand

$$K(p_1, \dots, p_N; v_1, \dots, u_N; a) = \sum_{1 \le j \le N} |a^{-1/2} v_j - a^{1/2} p_j|^2.$$

While zero minimum is achieved at a solution to the inverse problem, in this form the functional is not quadratic, which complicates applications of results of the calculus of variations. A first step to simplify the problem is to minimize over $a$, replacing in the above variational problem the integrand of the functional by

$$K_1(p_1, \dots, p_N; v_1, \dots, u_N) = \min K(p_1, \dots, p_N; v_1, \dots, u_N; a)$$

over $a \in [C_1, C_2]$.

In the plane case, where we assume that $\Omega$ is a simply connected domain, one can rewrite the condition that the $v_j$ are divergent free vector fields by representing them as $(\nabla w_j)^\perp = (-\partial_2 w_j, \partial_1 w_j)$ for some functions $w_j$ in $H_{(1)}(\Omega)$. Since $g_{1,j}$ is the boundary value of tangential derivatives of these functions, we are given their boundary values $H_{\bullet j}$ determined up to constants. When $C_1 = 0$, $C_2 = \infty$, it is easy to calculate

$$K_1(\nabla u_1, \dots, \nabla u_N; \nabla^\perp w_1, \dots, \nabla^\perp w_N)$$
$$= 2\left(\sum_{1 \le j \le N} |\nabla u_j|^2\right)^{1/2} \left(\sum_{1 \le j \le N} |\nabla w_j|^2\right)^{1/2} + 2 \sum_{1 \le j \le N} \det(\nabla u_j, \nabla w_j).$$

This integrand is not convex with respect to $\nabla u_j, \nabla w_j$ (because the function $|p_{11} q_{11}|$ obtained by letting $\nabla u_1 = p_1 = (p_{11}, 0)$, $\nabla w_1 = q_1 = (q_{11}, 0)$, $\nabla u_j = \nabla w_j = 0, 2 \le j$, is not a convex function of $p_{11}, q_{11}$). Kohn and Vogelius calculated the integrand of the relaxation and found that it is equal to

$$2\Bigg(\sum_{1\le j,k\le N}(\det(\nabla u_j,\nabla w_k))^{1/2}$$
$$+2\Bigg(\sum_{j<k,l<m}((\det(\nabla u_j,\nabla u_k))^2(\det(\nabla w_l,\nabla w_m))^2)^{1/2}\Bigg)^{1/2}$$
$$+2\sum_{1\le j\le N}\det(\nabla u_j,\nabla w_j)\Bigg).$$

It appears that the relaxed minimization problem needs regularization, which has not been discussed yet. So a theoretical foundation of this method is not quite complete, and neither are rigorous numerical tests. However, there is some practical numerical justification of its efficiency discussed in [KoV3], and some tests have been implemented by Kohn and McKenney [KoM]. Certainly, this algorithm looks very attractive from a mathematical point of view.

Klibanov [KlT] proposed to use convex cost functional based on the norm

$$\left(\int_\Omega e^{2\tau\varphi}|u|^2\right)^{1/2}$$

with large parameter $\tau$. In particular, he considered problem of finding speed of propagation in hyperbolic equations from lateral Cauchy data. So far convexity of this functional is proven only for finite-dimensional (Galerkin-type) approximations of the inverse problem, and a value of $\tau$, which guarantees convexity, grows with dimension of approximation space.

## 10.4 Layer-stripping

Let $\Lambda(t)$ be the Dirichlet-to-Neumann operator for the conductivity equation

(10.4.1) $$\operatorname{div}(a\nabla u)=0 \text{ in } \Omega(t),$$

where $\Omega(t)$ is the ball $B(0;t)$ in $\mathbb{R}^n$, $n=2,3$. We will derive an (operator) differential equation for $\Lambda$ as a function of $t$, assuming that $a\in\overline{\Omega(1)}$ and $t<1$. In polar coordinates $(r,\sigma)$, $|\sigma|=1$, equation (10.4.1) reads

(10.4.2) $$\partial_r a\partial_r u + r^{-1}(n-1)\partial_r u + r^{-2}\operatorname*{div}_{\sigma} a\nabla_\sigma u = 0,$$

where $\operatorname{div}_\sigma$ and $\nabla_\sigma$ are respectively divergence and gradient on the unit sphere. One can consult the book of Courant and Hilbert for $n=2,3$. In particular, we have the polar form of the plane conductivity equation

$$\partial_r a\partial_r u + r^{-1}\partial_r u + r^{-1}\partial_\theta a\partial_\theta u = 0.$$

By using the substitution $r=tr^*$, which reduces the Dirichlet problem in $B(0;t)$ to that in $B(0;1)$ and known results (e.g., given in the paper of Agmon, Douglis, and Nirenberg [ADN]) about continuity and differentiability of elliptic boundary problems with respect to a parameter, one can conclude that the solution $u(r, \sigma; t)$ to the conductivity equation (10.4.1) with the Dirichlet boundary condition

$$u(t, \sigma; t) = g_0(\sigma) \in H_{(3/2)}(\Sigma) \tag{10.4.3}$$

has the derivatives $\partial_t u \in H_{(2)}(\Omega(t))$. We will consider the Dirichlet-to-Neumann map represented as

$$\Lambda(t) g_0(\sigma) = (a\partial_r u)(t, \sigma; t). \tag{10.4.4}$$

Differentiating the boundary condition, we obtain

$$\partial_r u + \partial_t u = 0. \tag{10.4.5}$$

To obtain a differential equation for $\Lambda(t)$ we differentiate (10.4.4) with respect to $t$ by using the chain rule to obtain

$$\begin{aligned} \partial_t \Lambda(t) g &= (\partial_r(a\partial_r u))(t, \sigma, t) + (a\partial_t \partial_r u)(t, \sigma, t) \\ &= -r^{-1}(n-1)a\partial_r u - r^{-2} \operatorname*{div}_\sigma a\nabla_\sigma u + (a\partial_t\partial_r)(t, \sigma, t), \end{aligned}$$

where we make use of the conductivity equation (10.4.2). Since $a$ does not depend on $t$, we can differentiate equation (10.4.2) with respect to $t$ to conclude that $\partial_t u$ also solves this equation. Therefore, $a\partial_r \partial_t u = \Lambda \partial_t u$ on $\partial\Omega(t)$. Using that due to continuity $\partial_t \partial_r u = \partial_r \partial_t u$ and that according to (10.4.5) $\partial_t u = -\partial_r u = -a^{-1}\Lambda g_0$ on $\partial\Omega(t)$, we have

$$\partial_t \Lambda(t) g_0 = -r^{-1}(n-1)\Lambda(t) g - r^{-2} \operatorname*{div}_\sigma g_0 - \Lambda(a^{-1}\Lambda g_0).$$

After a little rearrangement we arrive at the differential operator equation of Riccati type for the Dirichlet-to-Neumann map

$$\partial_t \Lambda(t) + \Lambda(a^{-1}\Lambda) + r^{-1}(n-1)\Lambda + r^{-2} \operatorname*{div}_\sigma(a\nabla_\sigma) = 0. \tag{10.4.6}$$

Sylvester [Sy2] in a similar way derived this equation when $n = 2$, and he also showed that in fact, any operator function satisfying (10.4.6) and some natural "initial condition" at $t = 0$ is the Dirichlet-to-Neumann map for the conductivity equation (10.4.1). In the inverse problem we are given $\Lambda(1)$, which plays the role of the final data. Observe that $a$ is unknown, but it can be found on $\partial\Omega(t)$ by using boundary reconstruction, described in Section 5.1. In particular, Nachman's formula says that

$$a(t\sigma) = 2 \lim e^{-ir\sigma\cdot\xi} S\Lambda(t) e^{ir\sigma\cdot\xi} \text{ as } |\xi| \to \infty.$$

If we consider $\Lambda$ as an operator from $H_{(3/2)}$ onto $H_{(1/2)}$, and since the single layer potential operator is continuous from $H_{(1/2)}$ into $H_{(3/2)}$, we can conclude that the operator $\Lambda(t) \to a$ on $\partial\Omega(t)$ has certain continuity properties that have not been studied in detail. Thus, we can consider (10.4.6) as an evolution equation for $\Lambda(t)$ with the given final data, and we can try to solve this equation. The idea can be realized as a *layer-stripping* algorithm, which consists in the subsequent reconstruction of the conductivity coefficient $a$ layer by layer, starting from $r = 1$ and penetrating inside by progressing from the layer $\{t_j < r < t_{j-1}\}$ to the layer

$\{t_{j+1} < r < t_j\}$. Methods of propagation could be different, and we describe one of them a bit later. However, the basic underlying idea is that the Dirichlet-to-Neumann map satisfies some equation of evolution and can be found from this equation and the given final conditions. In the most naive form, one can try to use the Euler method from ordinary differential equations to discover immediately that one has a convergence problem. The Riccati equation (10.4.6) has some similarities with the backward heat equation discussed in Section 3.1, but there are some striking differences. First, equation (10.4.6) is nonlinear and second, it is not a scalar equation and even not a system of them, but an operator equation. Not much is known about such "backward" operator equations, neither theoretically nor numerically. One exception, is the result of Sylvester [Sy2] about the rotationally symmetric inverse conductivity problem. We briefly outline his main findings.

When $a = a(r), n = 2$, by separation of variables we have $\Lambda(t)e^{im\theta} = \lambda_m(t)e^{im\theta}$, and equation (10.4.6) becomes

$$\partial_t \lambda_m + a^{-1}(t)\lambda_m^2 + t^{-1}\lambda_m - t^{-2}a(t)m^2 = 0, \quad 0 < t < 1, m = 1, 2, \dots,$$

we are given the final data $\lambda_m(1)$, and one can show that we have the initial condition $\lambda_m(0) = ma(0)$. In [Sy2] it was shown that the symmetric matrices $((\lambda_j/j = \lambda_k/k)/(j+k))$ of any size must be positive definite, which guarantees existence of a unique analytic function $\lambda(r, z)$ on the right half-plane that maps this half-plane into itself and satisfies the condition $\lambda_m(r)/m = \lambda(r, m)$. then the Pick-Nevanlinna construction explicitly gives $\lambda(r, z)$ from $\lambda(1, z)$, and $a$ is recovered from this function by a trace formula. It turns out that to identify $a(r)$ uniquely one needs any sequence $\lambda_{m(j)}(1)$ such that the series $\sum m^{-1}(j)$ is divergent. Using this technique, Sylvester showed that the layer-stripping algorithm is convergent in the radial case when $n = 2$.

In the general plane case, layer-stripping was developed and tested numerically by Cheney, Isaacsons, and Somersalo [SoCII], though their paper does not contain a convergence proof or stability analysis of this possibly promising method. In particular, they used the Riccati equation for the inverse of the Dirichlet-to-Neumann map $\Lambda^{-1}$, which is the Neumann-to-Dirichlet map. This had certain advantages, theoretical (it is a smoothing operator) and practical (it is more convenient to prescribe currents $a\partial_\nu u$ and measure voltages $u$ on $\partial\Omega$). The new Riccati equation is

$$\partial_t \Lambda^{-1} - a^{-1} - r^{-1}(n-1) - r^{-2}\Lambda^{-1} \operatorname*{div}_{\sigma}(a\nabla_\sigma \Lambda^{-1}) = 0, \tag{10.4.7}$$

and it can be obtained from equation (10.4.6) by using the identity $\partial_t \Lambda = -\Lambda(\partial_t \Lambda^{-1})\Lambda$. The layer-stripping algorithm suggested in [SoCII] makes use of the trigonometric basis $\{e^{im\theta}\}$. The authors divide the interval [0, 1] into subintervals by the points $t_j$, $t_N < \cdots < t_1 < t_0 = 1$, and accordingly break down the unit disk $\Omega$ into $N$ disjoint annuli $\Omega_j = \{t_{j+1} < r \le t_j\}$. The key step of the algorithm consists in a determination of $a$ at $\{r = t_j\}$ and the propagation of $\Lambda$ into $\Omega_j$. The boundary reconstruction has been done by using the formula

$$a_k^{-1} = \pi^{-1} \lim |m| \int_0^{2\pi} e^{i(n+m)\theta} \Lambda^{-1}(t_j) e^{-im\theta} d\theta,$$

where the $a_k$ are the Fourier coefficients of the resistivity $a^{-1}$,

$$a_k^{-1} = \int_0^{2\pi} e^{-ik\theta} a^{-1}(t_j, \theta) d\theta.$$

Then, approximating $a^{-1}$ by finite Fourier sums and taking the inverse, one obtains an approximation of $a$ on the circle $\{r = t_2\}$. An approximation of the discretization of the operator $\Lambda^{-1}(t - j + 1)$ is obtained by using the Riccati equation (10.4.7), which is discretized. So far, the numerical test of [SoCII] have shown worse resolution and stability then those obtained by simple linearization.

Summarizing, we can tell that the layer-stripping method has obvious theoretical and computational advantages. In particular, it is applied to exact nonlinear inverse problems, and it has the property of locality, because during reconstruction one moves from $\partial\Omega$ to inside layer by layer, so one has no problems with local minima or limited memory, as in variational methods. On the other hand, despite all promises, at present there have been no theoretical results or good numerics obtained by this method.

## 10.5 Range test algorithms

Historically, the first range test algorithm, called *linear sampling method* was proposed by Colton and Kirsch [CoK] for numerical solution of inverse obstacle scattering problems.

To describe this method for scattering by hard obstacles we remind that the function

$$K(x - a; k) = e^{ik|x-a|}/(4\pi|x - a|) = e^{ik|x|}/(4\pi|x|) e^{-ik\sigma(x)\cdot a}(1 + O(|x|^{-1}),$$

so $\mathcal{A}(\sigma(x); k, a) = e^{-ik\sigma(x)\cdot a}$, $\sigma(x) = |x|^{-1}x$, is the scattering pattern of $K(x - a; k)$. We denote $e(\sigma; a) = -e^{-ik\sigma\cdot a}$. Let $G$ be the operator mapping Neumann boundary data of the exterior problem for the Helmholtz equation into the scattering pattern of its solution. In more detail, let $v$ be the solution of the exterior problem

(10.5.1) $$(\Delta + k^2)v = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D},\ \partial_\nu v(x) = g_1 \text{ on } \partial D$$

with the Sommerfeld radiation condition at infinity. For any $g_1 \in H_{(-1/2)}(\partial D)$ this solution exists and is unique, as described in section 6.0 (see also [CoKr], [McL]). Let $\mathcal{A}(; g_1)$ be the be the scattering pattern of $v$ and $Gg_1 = \mathcal{A}(; g_1)$. We remind that the Herglotz operator $H : L_2(S^2) \to H_{(-1/2)}(\partial D)$ is given by the formula

(10.5.2) $$H\phi(x) = \int_{S^2} exp(ik\sigma \cdot x)\phi(\sigma) dS(\sigma).$$

Finally, let $A$ be the scattering operator

(10.5.3) $$A\phi(\sigma) = \int_{S^2} \mathcal{A}(\sigma, \xi; k)\phi(\xi) dS(\xi), \; A : L_2(S^2) \to L_2(S^2).$$

Obviously,

(10.5.4) $$A = GH.$$

A simple observation is that

(10.5.5) $$a \in D \text{ if and only if } e(;a) \in \mathcal{R}G.$$

Indeed, $e(;a) \in \mathcal{R}G$ if and only if this function is the scattering pattern of the (scattering) solution $v$ to the exterior problem (10.5.1). Since scattering pattern uniquely determines scattering solution, this is equivalent to equality $v = K(-a; k)$ outside some large ball. Comparing singularities of both sides we conclude that this equality is possible only if $a \in D$. The observation (10.5.5) and the factorization (10.5.4) suggest the following method to decide whether $a \in D$. Solve the equation

(10.5.6) $$A\phi(;a) = e(;a)$$

for $\phi(;a)$. If the norm of $\phi(;a)$ is small, then $a \in D$, if is is large, then $a$ is outside $D$.

Unfortunately this naive approach has flaws. Firstly, the closure of range $\mathcal{R}H$ of $H$ is $H_{(-1/2)}(\partial D)$ if and only if $k^2$ is not the Dirichlet eigenvalue for the Laplace equation in $D$. More importantly, the Herglotz operator is highly smoothing, so its range never is $H_{(-1/2)}(\Gamma)$, hence it is certainly wrong that range of $A$ (which is known) coincides with range of $G$, that by (10.5.5) completely characterizes the obstacle $D$. So at best one can try to solve the equation (10.5.6) approximately, by using some regularization, and then there is a question how to modify (10.5.5) to decide whether $a \in D$.

To resolve difficulties with a straightforward approach, Kirsch suggested the following more detailed and complicated factorization of the far field operator $A$:

(10.5.7) $$A = CGS^*G^*$$

where $C$ is some constant and $S$ is the single layer operator

$$Sg(x) = \int_\Gamma K(x-y;k) g(y) d\Gamma(y)$$

considered from $H_{(-1/2)}(\Gamma)$ into $H_{(1/2)}(\Gamma)$. To be convinced that (10.5.7) holds we observe that using that $e^{-ik\sigma(x)\cdot y}$ is scattering pattern of $K(x-y;k)$ and the definitons of $S$ and $G$ we yield $GS = 4\pi H^*$. Now (10.5.7) follows from (10.5.3). The operator $S$ is an isomorphism between $H_{(-1/2)}(\partial D)$ and $H_{(1/2)}(\partial D)$. Using this property as well as the singular value decompositions for $G$ and the Picard's test (Lemma 2.3.5) Kirsch [Kir] showed that

$$\mathcal{R}G = \mathcal{R}(A_D^* A_D)^{1/4}.$$

Since the operator $A_D$ is given this equality combined with (10.5.5) gives a range test for detecting $D$ provided that $k^2$ is not a Dirichlet eigenvalue for $D$.

Under the same restriction on Dirichlet eigenvalues and using similar tools, Arens [Ar] partially justified convergence of some regularized linear sampling methods. In more detail, let $(\lambda_m, a_m, b_m)$ be the singular system for $A$ as defined in section 2.3. We define

$$R_\alpha \phi = \sum_m r(\alpha, \lambda_m)/\lambda_m (\phi, b_m)_2(S^2) a_m, \ \ r(\alpha, \lambda) = \lambda^2/(\alpha + \lambda^2).$$

It is not hard to see that $R_\alpha$ is a continuous linear operator from $L_2(S^2)$ into itself and it is a (Tikhonov) regularizer for the operator $A$. Using (10.5.4) it is shown in [Ar] that $HR_\alpha$ is a regularizer for the operator $G$. Hence if $a$ is not in $D$, we have $\|HR_\alpha e(;a)\|_{(1/2)}(\partial D) \to \infty$ as $\alpha \to 0$, and by coninuity of $H$ it follows that $\|R_\alpha e(;a)\|_2(S^2) \to \infty$ as $\alpha \to 0$. Since $A$ and $R_\alpha$ are determined by scattering data, then boundedness of $\|R_\alpha e(;a)\|_2(S^2)$ implies that $a \in D$. At the author's knowledge, this is the best result toward justification of the linear sampling method.

Despite lack of complete theoretical base, the linear sampling method proved to be quite successful in numerical solution of inverse scattering problems. It is linear, does not involve any assumption about number of components of $D$, avoids expensive numerical solution of the direct scattering problem required by iterative Newton type methods, and is easily implemented numerically. This method can be adjusted to a variety of important applied problems, in particular, to inverse scattering of electromagnetic waves described by the Maxwell's system. Indeed, Colton, Haddar, and Piana [CoHP] obtained recognizable images of an aircraft from electromagnetic scattering data at typical frequencies collected from all incident scattering directions. The modification of this method by Kirsch is better justified theoretically, but the method is more complicated and does nor produce better numerical results than original linear sampling. One of origin of the linear sampling methods were the author's papers [Is3], [Is5] on use of singular solutions for proofs of uniqueness of identification of transparent obstacles from the Dirichlet-to-Neumann map and scattering data and their continuation by Kirsch and Kress [KirK] for hard obstacles. It seems that for both theoretical and numerical purposes a singular solution of Green's function type is the best. Probably, this is due to combination of (local) positivity of real parts and singularity. Use of higher singularities (like derivatives of Green's function) in numerical experiments did not show any improvement. On theoretical side, say for uniqueness or stability proofs, the linear sampling method and its modifications produce more limited results than the original method of singular solutions. Typically, they are applicable under conditions that $k^2$ is not an eigenvalue of certain elliptic boundary value problem in $D$. However, these elliptic problems are known to have many eigenvalues in interesting ranges of $k$.

The factorization method of Kirsch was used by Hanke and Brühl [HB] in numerical reconstruction of an hard obstacle from the Neumann-to-Dirichlet map for the Laplace equation and by Kirsch in the inverse conductivity problem to identify dscontinuity surface of anisotropic conductivity. We will briefly describe a simpler problem from [HB]. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with $C^2$-boundary and let $D$ be an open subset of $\Omega$ with $C^2$-boundary. Let $\Lambda^{-1}$ be the Neumann-to-Dirichlet map determined by $D$ in the following way. As knowm, for any

$$g_1 \in H_{(-1/2)}(\partial\Omega), \quad \int_{\partial\Omega} g_1 d\Gamma = 0$$

there is a unique solution $u \in H_{(1)}(\Omega \backslash \overline{D})$ to the Neumann problem

$$\Delta u = 0 \text{ in } \Omega\backslash\overline{D}, \ \partial_\nu u = g_1 \text{ on } \partial\Omega, \ \partial_\nu u = 0 \text{ on } \partial D$$

such that the integral of $u$ over $\partial\Omega$ is zero. We define $\Lambda^{-1} g_1 = u$ on $\partial\Omega$. Let $\Lambda_0^{-1}$ be the Neumann-to- Dirichlet map for void $D$. Let $E(;a)$ be the $x_1$ derivative of the Neumann function for the Laplace operator in $\Omega$. As known, $E(x;a) = C(x_1 - a_1)|x - a|^{-n} + v(x;a)$, where $v$ is bounded for fixed $a \in \Omega\backslash\overline{D}$. From known results (similar to Theorem 4.1) it follows that $\Lambda$ is a continuous operator from its domain in $H_{(-1/2),0}(\partial\Omega)$ into $H_{(1/2),0}(\partial\Omega)$ and that $(\Lambda - \Lambda_0)$ is positive (in $L_2$). Here additional index 0 denotes subspaces of functions with zero integral over $\partial\Omega$. Let $G_D : H_{(-1/2,0)}(\partial D) \to H_{1/2,0}(\partial\Omega)$ be the Green's operator mapping the Neumann data $h_1$ on $\partial D$ for the Neumann problem:

$$\Delta v = 0 \text{ in } \Omega\backslash\overline{D}, \ \partial_\nu v = h_1 \text{ on } \partial D, \ \partial_\nu v = 0 \text{ on } \partial\Omega$$

into the Dirichlet data $v$ on $\partial\Omega$. As above, it can be demonstrated that

$$\Lambda - \Lambda_0 = G_D S_1 G_D^*$$

where $S_1$ is an isomorphism from $H_{(1/2),0}(\partial D)$ onto $H_{(-1/2),0)}(\partial D)$.

Summing up it, as shown in detail in [HB],

$$a \in D \ \text{ if and only if } \ E(;a) \in \mathcal{R}(\Lambda - \Lambda_0)^{1/2}$$

which suggests the range test similar to the factorization method in the scattering situation. This range test was suggested and numerically implemented in [HB] with satisfactory results.

In case of many boundary measurements there are similar algorithms proposed and implemented numerically by Ikehata [Ik] (probe method) and Potthast [Po] (singular sources method). These methods also have some origin in the method of singular solutions, they are theoretically justified and some cases proved to be competitive with the linear sampling method.

Belishev [Be3] suggested a use of the so-called "mark function" (singular solution of the Laplace equation) to find speed of the propagation in inverse hyperbolic problems with given lateral (local) Dirichlet-to Neumann map. Indeed, the result of Exercise 8.4.3 can be easily extended onto fucntions $v$ which are harmonic not in the whole $\Omega$ but only near the subdomain $\Omega_{2T}$ filled by waves from $\Gamma_0$ in time $T$. Then one can use as $v$ a harmonic function with singuarity at $a$ to decide whether $a \in \Omega_T$ and hence to determine the spped of propagation. This idea can lead to an efficient range type test in inverse hyperbolic problems.

One of recent range tests due to Kusiak and Sylvester [KuS] and to Potthast, Sylvester, and Kusiak [PoSK] checks whether a domain $D$ with connected $\mathbb{R}^3\setminus\overline{D}$ contains singularities of a radiating solution $v$ to the Helmholtz equation. This test applies to single or many boundary measurements or scattering data. Let $\mathcal{A}_v(\sigma; k)$ be the scattering pattern of a radiating solution $v$ to the Helmholtz equation. Let $k^2$ be not a Dirichlet eigenvalue of the Laplace operator for $D$. The main observation is that the equation

$$\int_{\partial D} e^{ik\sigma(x)\cdot y} g(y) d\Gamma(y) = \mathcal{A}_v(\sigma(x); k) \tag{10.5.8}$$

has a solution $g \in H_{(1/2)}(\partial D)$ if and only if

$$\int_{\partial D} K(x-y; k) g(y) d\Gamma(y) = v(x), \;\; |x| > R$$

or, equivalently, if and only if

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^3\setminus\overline{D}, \;\; v \in H_{(1)}(B \setminus \overline{D}), \tag{10.5.9}$$

for some large ball $B$ and $v$ satisfies the Sommerfeld radiation condition. In other words, $D$ contains singularities of $v$ if and only if the scattering pattern $\mathcal{A}_v$ of $v$ is contained in the range of the operator $A_D : H_{(1/2)} \to L_2(S^2)$ defined by the left side of (10.5.8). In [KuS], [PoSK] they define the convex scattering support of a scattering amplitude of a radiating solution $v$ as the intersection of all convex domains $D$ such that (10.5.9) holds. In [KuS] they describe several properties of convex scattering support. In particular, there is an analogue of Paley-Wiener Theorem characterizing convex scattering support of $\mathcal{A}(\sigma; k)$ via the growth of the Fourier coefficients of in the plane case. The equation (10.5.8) is used in [PoSK] for numerical reconstruction of convex scattering support of $\mathcal{A}$. The operator from the left side of (10.5.8) is highly smoothing, so for efficient numerics one needs to regularize (10.5.8). The Tikhonov regularization is

$$\alpha g(;\alpha) + A_D^* A_A g(;\alpha) = A_D^* \mathcal{A}_v. \tag{10.5.10}$$

The range test proposed in [PoSK] consists of four steps: 1) choice of a family of convex domains $D$, 2) choice of (small) positive $\alpha$ and $C$, 3) solution of (10.5.10) for each $D$ and computation of $\|g(;\alpha)\|_2(\partial D)$, 4) finding the intersection of all test domains $D$ for which this norm is less than $C$. If one use one set of scattering data in inverse obstacle problem, like a scattered wave from one incident direction, then generally convex scattering support is strictly inside an obstacle $D$ and one can not determine $D$. For example, if $D$ is a soft (or hard obstacle) with analytic boundary, then by known results ([Mor], section 6.7) $v$ analytically continues across $\partial D$ inside $D$. However, if one uses a complete set of incident waves, singularities produced by limits of linear combinations of scattering solutions fill out $D$ which will be recovered completely. In [PoSK] there are examples of satisfactory numerical reconstruction of soft and hard scatterers of medium size from one set of scattering data (one incident direction) and with 1% noise.

An important question arising in particular in this method is about singularities of (radiating) solutions $v$ to the Helmholtz equation satisfying Dirichlet or Neumann boundary conditions at $\partial D$ or generated by certain source $f$. This question is briefly discussed in [KuS] for the Helmholtz equation in the plane and in the book [Is4], chapter 4, where the main focus is on singularities of volume potentials of domains. The question about singularities of solutions of exterior classical boundary value problems for the Helmholtz equationis largely open. In particular, it is interesting to get sufficient conditions to guarantee singular behavior of solutions to exterior problems at corners of polyhedrons in $\mathbb{R}^3$ and therefore to create a solid theoretical background for numerical recovery of polyhedrons from scattering data via range type tests.

Another important ( and mainly open) question is about stabililty and rates of convergence of range type algorithms. Only proof of convergence hardly justifies an algorithm, because it is important how fast it is convergent and how sensitive is it to errors in data. The problems discussed in this section are severely ill-posed, however there are observations that in some cases stability and resolution increase with frequency $k$, while in some other cases stability is decreasing with $k$. The study of this phenomena was started by Hrycak and Isakov [HrI] as described in section 3.4, and further progress can lead to very efficient and robust numerical algorithms for important inverse problems.

## 10.6 Discrete methods

One can attempt to solve an inverse problem for a partial differential equation by discretizing the differential equation and trying to solve the discretized inverse problem. This general scheme works in some cases, but discretization and solving of discretized problems is not a simple method. We will describe here some results of Curtis and Morrow [CuM1], [CuM2] for a discretized version of the conductivity equation. To describe this version, we consider an $N \times N$ grid $\Omega_N$ on the unit square $\Omega$ formed by points $x(j,k) = (j/N, k/N), 0 \le j \le N, 0 \le k \le N$. We let $u_{jk} = u(x_{jk})$. We will consider the following conductivity equation:

$$\sum_{l,m} a_{j+l,k+m}(u_{j+l,k+m} - u_{jk}) = 0 \quad \text{on } \Omega_N. \tag{10.6.1}$$

This equation is a discretization of the conductivity equation $a\Delta u + \nabla a \cdot \nabla u = 0$, where we replace partial derivatives by standard finite differences. Solutions to equation (10.6.1) are called discrete harmonic functions. They have properties similar to those of harmonic functions; in particular, they satisfy the maximum principle, and it is easy to prove that one can uniquely solve the discrete Dirichlet problem that augments equation (10.5.1) by the boundary condition

$$u_{jk} = g_{0,jk} \text{ when } j = 0, N \text{ or } k = 0, N. \tag{10.6.2}$$

In the discrete case there is no problem to define the Neumann data

$$a_{jk10}(u_{j+1,k} - u_{j,k}) = g_{1,jk} \text{ when } j = 0, N-1,$$
$$a_{jk01}(u_{j,k+1} - u_{j,k}) = g_{1,jk} \text{ when } k = 0, N-1. \tag{10.6.3}$$

The discrete Dirichlet-to-Neumann map $\Lambda_N$ maps the $4N$-dimensional vector of the discrete Dirichlet boundary data $(g_{0,1}, \dots, g_{0,N})$ into the $4N$-dimensional vector of the discrete Neumann data $(g_{1,1}, \dots, g_{1,N})$. Here the boundary data $g_{0,jk}$ are labeled by coordinates $g_{0,1}, \dots, g_{0,N}$ starting with $g_{0,1} = g_{0,N,N-1}$ clockwise, so that the last coordinate is $g_{0,4N} = g_{0,N-1,N}$. The coordinates $g_{1,j}$ similarly label the discrete Neumann data. Accordingly, $\Lambda_N$ is represented in these coordinates by the $4N \times 4N$ matrix $(\lambda_{jk})$. It is not difficult to see that this matrix is symmetric. Another property of this matrix that is proven by Curtis and Morrow is that for any $m \leq N$ there are numbers $\alpha_1, \dots, \alpha_m$ such that

$$\lambda_{j,4N-m+1} + \sum_{1 \leq l \leq m} \lambda_{jk}\alpha_k = 0 \text{ when } m \leq j \leq 4N - m. \tag{10.6.4}$$

This property is the key for the reconstruction procedure and for characterization of the discrete Dirichlet-to-Neumann map.

The reconstruction starts with the right upper corner of $\Omega_N$. To find $a_{N-1,1}$ and $a_{1,N-1}$ we will make use of the relation (10.6.4) with $m = 1$, which claims that there is $\alpha_1$ such that $\lambda_{j,4N} + \lambda_{j,1}\alpha_1 = 0$ holds for all $j = 2, \dots, 4N-1$. The value of $\alpha_1$ can be found from this relation with, say, $j = 2$. Let the Dirichlet boundary data be $g_{0,1} = \alpha_1$, $g_{0,4N} = 1$, and other $g_{0,j} = 0$. Then the solution of the discrete Dirichlet problem $u$ is zero at all interior points of $\Omega_N$; in particular, $u_{N-1,N-1} = 0$. By the definition of the Dirichlet-to-Neumann map we have the Neumann data

$$g_{1,1} = \lambda_{1,4N} + \lambda_{1,1}\alpha_1 \text{ and } g_{1,4N} = \lambda_{4N,4N} + \lambda_{4N,1}\alpha_1.$$

Since we know these Neumann data and $u_{N,N-1} = \alpha_1$, $u_{N-1,N-1} = 0$, $u_{N-1,N} = 1$, the conductivities at the upper right corner points are

$$a_{N,N-1} = g_{1,1}/\alpha_1, \quad a_{N-1,N} = g_{1,4N}$$

When the conductivities above the $(m+1)$th diagonal from above $\{x_{j,2N-m-1-j}\}$ are calculated, one can use the relation (10.6.4) to find them on this diagonal. To solve the overdetermined system (10.6.4) for $\alpha_1, \dots, \alpha_m$, it suffices to consider only the equations with, say, $j = 3N - m + 1, \dots, 3N$ and to use that due to the determinant property (10.6.5,d), this smaller system is uniquely solvable. Then, as in the case $m = 0$, we consider the solution $u$ to the discrete Dirichlet problem with boundary data $g_{0,1} = \alpha_1, \dots, g_{0,m} = \alpha_m$, $g_{0,4N-m+1} = 1$, and $g_{0,j} = 0$ for other indices $j$. As above, $u = 0$ on or below the diagonal $\{x_{j,2N-m-1-j}\}$. Since the conductivity coefficient is already known at points where $u$ is not zero, we can uniquely determine $u$ by solving the Dirichlet problem for (10.6.1). Moving from the left point of the diagonal to the right point, we can subsequently determine the conductivity on the diagonal.

Moving from the upper right corner, one can uniquely determine $a_{j,k}$ when $j \leq k$. To determine other values one can similarly start from the lower left corner.

To formulate a characterization result, it is convenient to write $\Lambda_N$ as the block matrix with $N \times N$ matrix entries $\Lambda_{lm}$, $l, m = 1, \ldots, 4$. Curtis and Morrow proved that a $4N \times 4N$ matrix $(\lambda_{jk})$ represents a discrete Dirichlet-to-Neumann map if and only if

(a) It is symmetric.

(b) $\sum_{1 \leq k \leq 4N} \lambda_{jk} = 0$, $j = 1, \ldots, 4N$.

(c) The condition (10.6.4) is satisfied.

(10.6.5) (d) $\Lambda_{lm}$ has the determinant property when $l < m$.

A square matrix is said to have the determinant property if any of its square $k \times k$ submatrices has positive determinant when $k = 4m + 1$ or $4m + 2$ and negative determinant if $k = 4m + 3$ or $k = 4m$.

While the determinant condition (10.6.5) resembles Krein's positivity condition for the one-dimensional inverse hyperbolic (or spectral) problem described in the papers of Krein [Kr] or Symes [Sym], it has not yet been possible to utilize it, and at present there is no characterization result for the Dirichlet-to-Neumann map in the continuous case. In fact, at present there is no continuous analogue of conditions (10.6.5) (b) and (d). Furthermore, nobody has found a continuous analogue of the discrete reconstruction procedure described above.

Recently, Druskin and his collaborators [DrK] found optimal posititioning of grid points for finite differences versions of problems of continuation of geophysical fields. They are based on classical results from function and number theories and show significant improvements in particular in the problems with very limited amount of data. In our opinion, also optimizing positions of receivers (like microphones in acoustics) in many inverse problems has a promise for increase of resolution.

# Appendix

## Functional Spaces

We collect here definitions and known results about space $C^{k+\lambda}$ of Hölder continuous functions and about Sobolev space $H_{k,p}$ that we used in the book.

We recall that $C^{\lambda}(\overline{\Omega}), 0 < \lambda < 1$, consists of continuous functions $u$ on $\overline{\Omega}$ with finite norms $|u|_{\lambda}(\Omega) = |u|_0(\Omega) + \sup |u(x) - u(y)|/|x - y|^{\lambda}, x \neq y, x, y \in \Omega$. Here $\Omega$ is any subset of $\mathbb{R}^n$. The norm $|u|_0 = \sup |u(x)|, x \in \Omega$. The space $C^{k+\lambda}(\overline{\Omega})$ is formed of functions $u$ with finite norm $|u|_{k+\lambda}(\Omega) = \sum_{|\alpha| \leq k} |\partial^{\alpha} u|_{\lambda}(\Omega)$. When $\lambda = 0$, these spaces are defined when only the term $|u|_0$ is left in the definition of the norm. These are known to be Banach spaces.

Now we introduce Sobolev space $H_{k,p}(\Omega)$ for open sets $\Omega$ in $\mathbb{R}^n$ of for $C^k$-smooth manifolds. We recall that for $k = 0, 1, \ldots$ this space can be defined as the completion of $C^k(\overline{\Omega})$ with respect to the norm $\|u\|_{k,p}(\Omega) = (\sum \|\partial^{\alpha} u\|_p^p(\Omega))$, where the sum is over $|\alpha| \leq k$, and $\mathring{H}_{k,p}(\Omega)$ is the completion of $C_0^k(\Omega)$ with respect to the same norm. For negative $k$, the space $H_{k,p}(\Omega)$ is defined as the space of linear continuous functionals on $\mathring{H}_{k,p}(\Omega)$. It is known (see Lions and Magenes [LiM] for the case $p = 2$) that an element $u$ of such a space can be represented as $\sum \partial^{\alpha} u_{\alpha}$ with $u_{\alpha} \in L_p(\Omega)$, $|\alpha| \leq k$, and derivatives are understood in the weak sense.

**Theorem A1** (Extension). *For any set $\Omega$ in $\mathbb{R}^n$ there is a linear continuous operator $E$ mapping $C^{k+\lambda}(\overline{\Omega})$ into $C^{k+\lambda}(\mathbb{R}^n)$ such that $Eu = u$ on $\Omega$. This operator depends on $\Omega$, $k$, and $\lambda$, but its norm depends only on $k$, $\lambda$, and* diam $\Omega$.

*For any Lipschitz $\Omega \subset \overline{\Omega} \subset B(0; R)$ and $k = 0, 1, \ldots$ there is a continuous operator $E$ mapping $H_{k,p}(\Omega)$ into $\mathring{H}_{k,p}(B(0; R))$ such that $Eu = u$ on $\Omega$. If $\partial\Omega \in C^k$, then there is a similar continuous extension operator from $H_{(s)}(\Omega)$ into $\mathring{H}_{(s)}(\mathbb{R}^n)$ when $s \leq k$ and a bounded extension operator from $H_{(k-1/2)}(\partial\Omega) \times \ldots \times H_{(1/2)}(\partial\Omega)$ into $H_{(k)}(\Omega)$ such that the extended function $u$ has the given Cauchy data $(u, \ldots, \partial_{\nu}^{k-1} u)$ in this product of spaces.*

The result about Hölder spaces is a version of the Whitney extension theorem. Its proof (as well as a definition of $C^{k+\lambda}$ for general sets $\Omega$) can be found in the book of Stein [Ste], extension operators for Sobolev spaces are constructed in the book of

Morrey [Mor] for integer $k$, and the case $p = 2$ for all nonnegative $k$ is considered in the book of Lions and Magenes [LiM]. In [LiM] it is also observed that the operator extending $u$ as 0 outside $\Omega$ is continuous from $H_{(s)}(\Omega)$ into $H_{(s)}(\mathbb{R}^n)$ if and only if $0 \leq s < \frac{1}{2}$.

**Theorem A2** (Embedding). *For any bounded Lipschitz domain* $\Omega$ *there is a constant* $C(p, q, \lambda)$ *such that for all functions* $u \in H_{k,p}(\Omega)$ *we have*

$$\begin{aligned}
&\|u\|_q(\Omega) \leq C\|u\|_{k,p}(\Omega)\\
&\|u\|_{m,q}(\Omega) \leq C\|u\|_{k,p}(\Omega)\\
&|u|_\lambda(\Omega) \leq C\|u\|_{k,p}(\Omega)\\
&\qquad \textit{when } q \leq np/(n-kp), n > kp,\\
&\qquad \textit{when } m \leq k,\ p \leq q, n(1/p - 1/q) \leq k - m,\\
&\qquad \textit{when } \lambda \leq k - n/p, n < kp.
\end{aligned}$$

*Moreover, in case of strict inequalities corresponding embedding operators are compact.*

This is a classical result basically obtained by Sobolev in the 1930s. The second inequality is more contemporary and one can find it in many books on embedding and interpolation of Sobolev type spaces.

**Theorem A3** (On traces). *For any bounded Lipschitz domain* $\Omega \subset \mathbb{R}^n$ *and any* $(n-1)$*-dimensional Lipschitz surface* $S \subset \overline{\Omega}$ *there is a constant* $C(S, k, q, p)$ *such that for all functions* $u \in H_{k,p}(\Omega)$ *we have*

$$\|u\|_q(S) \leq C\|u\|_{k,p}(\Omega) \textit{ when } 1 < kp < n, q \leq p(n-1)/(n-pk),\ \ S \in C^k$$

$$\|u\|_{(1/2)}(S) + \|\nabla u\|_{(-1/2)}(S) \leq C\|u\|_{(1)}(\Omega).$$

The result for $H_{k,p}$-spaces can be found in [LU] while the claim about $H_{(k)}$-spaces is proven in the book of Lions and Magenes [LiM].

**Theorem A4** (Interpolation). *There is a constant* $C(\Omega)$ *such that*

$$\begin{aligned}
|\partial^\alpha u|_m u(\Omega) &\leq C|u|_{k+\lambda}^{(|\alpha|+\mu)/(k+\lambda)}(\Omega)|u|_0^{1-(|\alpha|+\mu)/(k+\lambda)}(\Omega),\\
\|u\|_{(s)}(\Omega) &\leq C\|u\|_{(s_1)}^{1-\theta}(\Omega)\|u\|_{(s_2)}^{\theta}(\Omega),
\end{aligned}$$

*provided that* $s = (1-\theta)s_1 + \theta s_2 \neq -\frac{1}{2} - k$, *for any* $k = 0, 1, 2, \ldots, 0 < \theta < 1$.

One can find a proof for spaces $H_{(s)}$ in the book of Lions and Magenes [LiM] and McLean [McL].

For the reader's convenience we recall also the integration by parts formula

$$\int_\Omega u\partial_j v = \int_{\partial\Omega} uv\nu_j d\Gamma - \int_\Omega \partial_j uv,$$

which is valid at least for functions $u \in H_{1,p}(\Omega)$, $v \in H_{1,q}(\Omega)$, $1/p + 1/q = 1$, $1 \leq p$, and domains $\Omega$ with piecewise Lipschitz boundary $\partial\Omega$. More exactly, $u, v$ (which are defined only almost everywhere) must be understood as their representatives with well-defined traces on $(n-1)$-dimensional surfaces. Integration by parts leads to the following formula for the (formally) adjoint $A^*$ to the differential operator $A = \operatorname{div}(a\nabla) + b \cdot \nabla + c$:

$$A^* = \operatorname{div}(a\nabla) - \overline{b} \cdot \nabla + (-\operatorname{div} \overline{b} + \overline{c})$$

and to the following Green's formula,

$$\int_\Omega (\overline{v} Au - u\overline{A^* v}) = \int_{\partial\Omega} ((\overline{v}\partial_{\nu(a)} u - u\partial_{\nu(a)}\overline{v} + (b \cdot \nu)u\overline{v}),$$

which takes the following simple form when $A = \Delta$:

$$\int_\Omega (v\Delta u - u\Delta v) = \int_{\partial\Omega} (v\partial_\nu u - u\partial_\nu v)\,.$$

Here $\partial_{\nu(a)} v = \nu \cdot a\nabla u$.

For a reader's convenience we remind that $\operatorname{curl}(u_1, u_2, u_3) = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)$ and that $\operatorname{curl}\operatorname{curl}\mathbf{u} = \nabla div\,\mathbf{u} - \Delta\mathbf{u}$. Also we give the Leibniz' formula [Hö2]

$$P(x, \partial)(uv) = \sum (P^{(\alpha)}(x, \partial)u)\partial^\alpha v/\alpha!$$

where $P(x, \partial)u = \sum a_\alpha(x)\partial^\alpha u$ is a linear partial differential operator of order $m$, the sums are over $|\alpha| \leq m$, $P(x, \xi) = \sum a_\alpha \xi^\alpha$ and $P^{(\alpha)}(\xi) = \partial_\xi^\alpha P(x, \xi)$.

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